

Minimal Problems in Computer Vision

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in collaboration with

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Oskarsson, Kalle Astrom, Alge Wallis, Martin Byrod,
Klas Josephson

Joe Kileel, Bernd Sturmfels

3D Reconstruction from Photographs



Capturing Reality (capturingreality.com)

Camera & structure computation essential ...

3D reconstruction

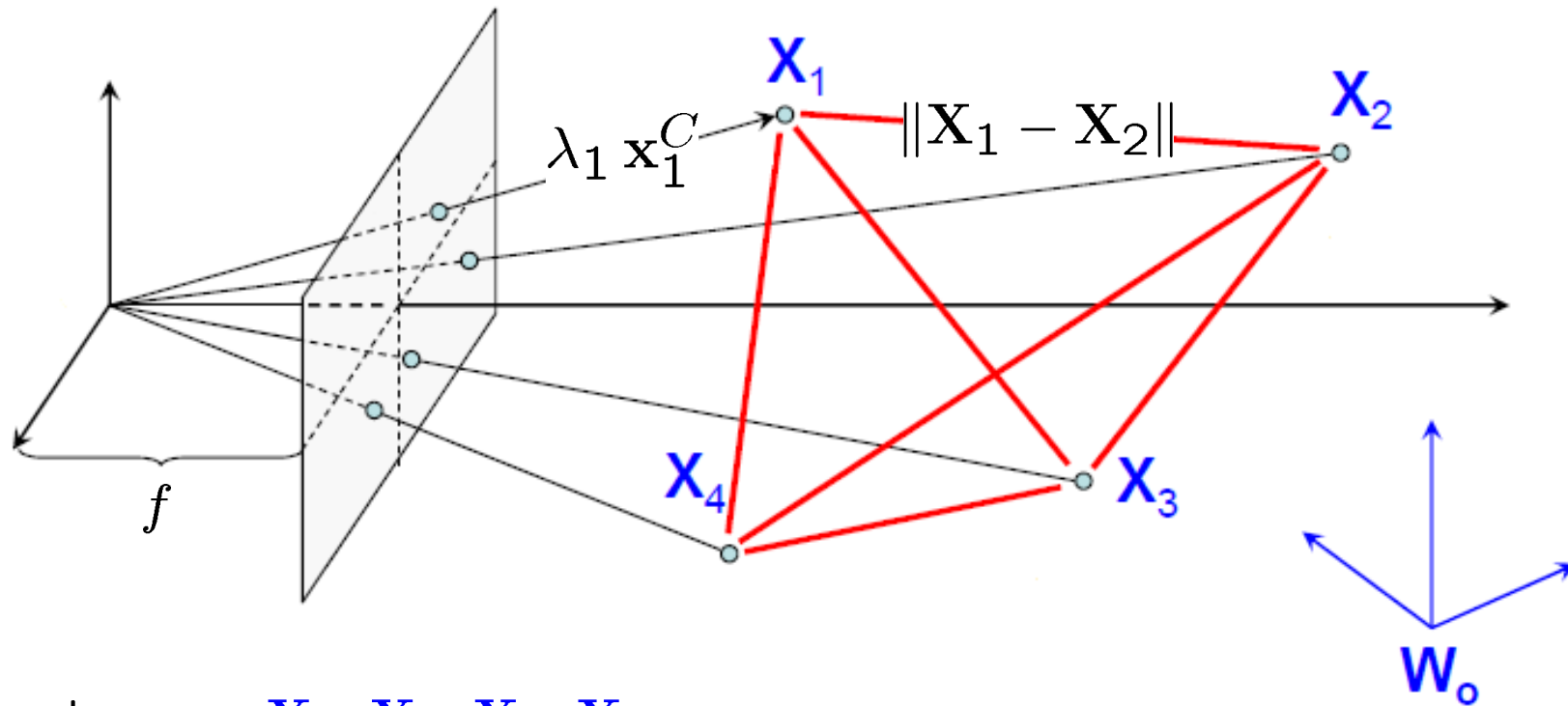


Solving "Minimal Problems"

by

Algebraic Geometry

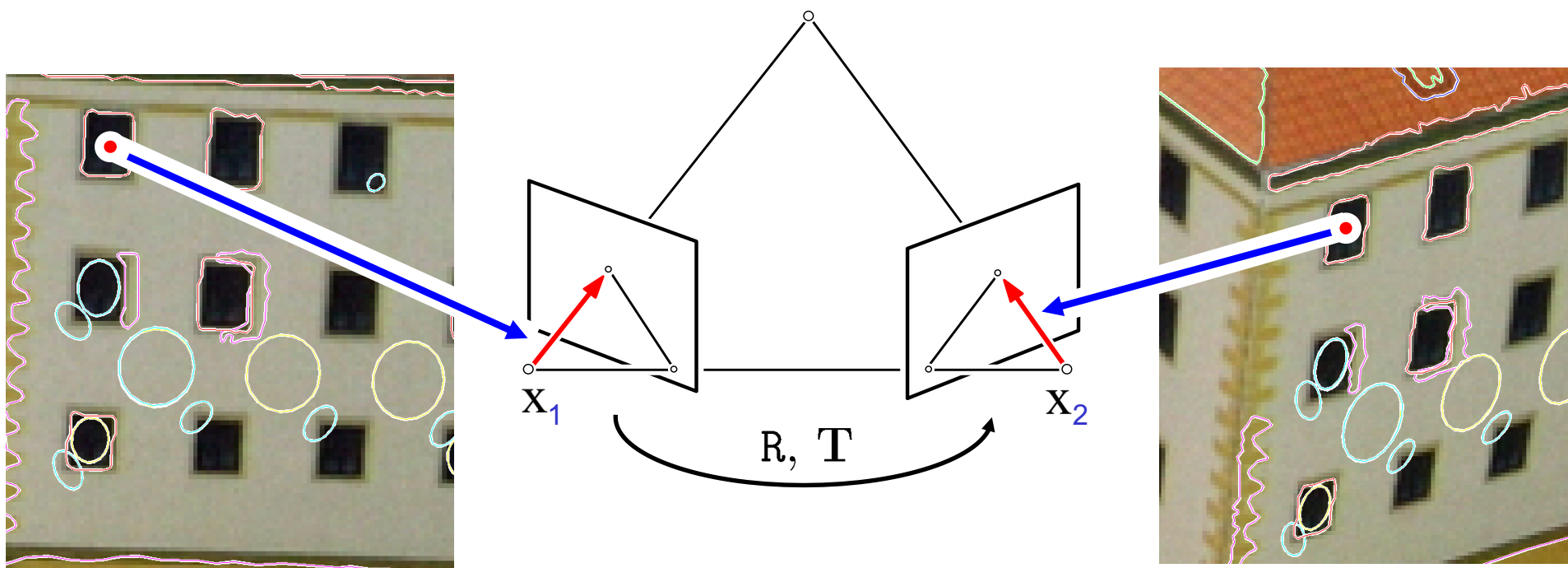
Minimal problem: Absolute Camera Orientation



- known $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$
- unknown $\mathbf{R}, \mathbf{T}, f$

$$\underbrace{\frac{\|\mathbf{X}_i - \mathbf{X}_j\|^2}{\|\mathbf{X}_k - \mathbf{X}_l\|^2} = \frac{\|\lambda_i \mathbf{x}_i^C - \lambda_j \mathbf{x}_j^C\|^2}{\|\lambda_k \mathbf{x}_k^C - \lambda_l \mathbf{x}_l^C\|^2}}_{\text{Algebraic equations}} \xrightarrow{\text{solve}} \lambda_i \longrightarrow \mathbf{R}, \mathbf{T}, f$$

Minimal problem: Relative Camera Orientation



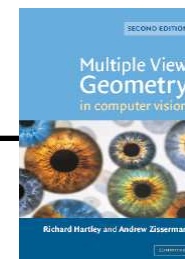
$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

$$\det \mathbf{F} = 0$$

$$2 \mathbf{F} \mathbf{F}^\top \mathbf{F} - \text{trace}(\mathbf{F} \mathbf{F}^\top) \mathbf{F} = 0$$

solve

\mathbf{F}

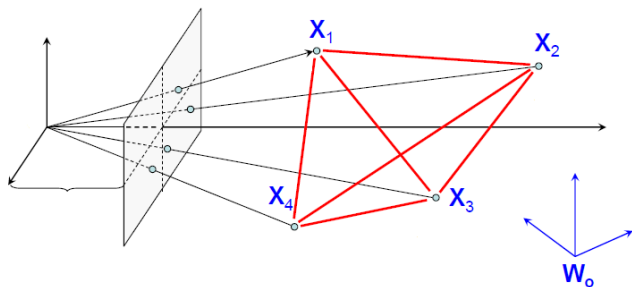


$\mathbf{R}, \frac{\mathbf{T}}{\|\mathbf{T}\|}$

Algebraic equations
from 5 correspondences

Long history of Minimal Problems

Absolute Camera Orientation



1841 J. A. Grunert

.

.

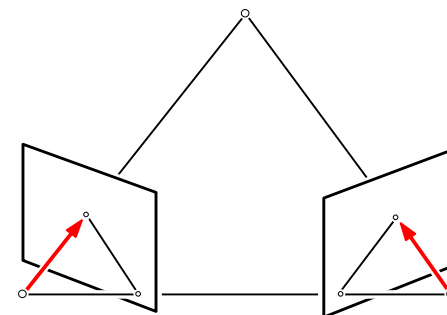
1981 M. A. Fischler and R. C. Bolles

.

.

2013 Z. Kukelova et al.

Relative Camera Orientation



1913 E. Kruppa

.

.

1981 H. Longuet-Higgins

.

.

2004 D. Nister

- many papers
- many applications

Minimal problems list (cmp.felk.cvut.cz/minimal)

← → ↺ cmp.felk.cvut.cz/mini/ ☆ ☰

Minimal Problems in Computer Vision

This page provides a list of papers, software, data, and evaluations for solving minimal problems in computer vision, which is concerned with finding parameters of (geometrical) models from as small (minimal) data sets by solving systems of algebraic equations.

Please send links to papers that should be listed here to Tomas Pajdla (pajdla@cmp.felk.cvut.cz) or Zuzana Kukelova (zukuke@microsoft.com).

Show entries

Title
1-Point-RANSAC Structure from Motion for Vehicle-Mounted Camera by Exploiting Non-holonomic Constraints
3D reconstruction from image collections with a single known focal length
A 4-Point Algorithm for Relative Pose Estimation of a Calibrated Camera with a Known Relative Rotation Angle
A Column-Pivoting Based Strategy for Monomial Ordering in Numerical Gröbner Basis Calculations
A Complete Characterization and Solution to the Microphone Position Self-Calibration Problem
A Direct Least-Squares (DLS) Method for PnP
A general solution to the P4P problem for camera with unknown focal length

• Unknown internal orientations

• Multi-camera systems

• Rolling shutter

• ...

1. New difficult problems

2. Systematic approach needed

... 12 more pages ...

• ~ 100 papers

• not complete ...

Minimal Problems in Computer Vision

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Show entries

Title	Authors	Published	Year	Links
1-Point-RANSAC Structure from Motion for Vehicle-Mounted Camera by Exploiting Non-holonomic Constraints	D. Scaramuzza	ICCV 95:74-85	2011	[pdf]
3D reconstruction from image collections with a single known focal length	M. Bujnak, Z. Kukelova, T. Pajdla	ICCV 2009, Kyoto, Japan, September 29 - October 2, 2009	2009	[pdf]
A 4-Point Algorithm for Relative Pose Estimation of a Calibrated Camera with a Known Relative Rotation Angle	R. Li, L. Heng, C. Pohlmann	IMOS 2013	2013	[pdf]
A Column-Pivoting Based Strategy for Monomial Ordering in Numerical Gröbner Basis Calculations	M. Byrd, K. Josephson, K. Astrom	ECCV 2008	2008	[pdf]
A Complete Characterization and Solution to the Microphone Position Self-Calibration Problem	Y. Kuang, S. Burgess, A. Tretter, K. Astrom	The 38th International Conference on Acoustics, Speech, and Signal Processing	2013	[pdf]
A Direct Least-Squares (DLS) Method for PnP	Heach, J.A. & Roumeliotis, S.I.	ICCV 2011	2011	[pdf]
A Minimal Solution to the	O. Sauer, M.			

Relative camera pose, radial distortion, focal length, GB [pdf] || Relative camera pose, radial distortion, focal length | [pdf] |
Relative camera pose, radial distortion, focal length, GB	[pdf]
Generalized P3P	[pdf]
Generalized P3P + unknown scale	[pdf]
Absolute camera	

View Camera

The Minimal Structure and Motion Problems with Missing Data for 4D Retina Vision

Authors: M. Oskarsson, K. Astrom, N.Chr. Overgaard

Published: Journal of Mathematical Imaging and Vision 2006 [http]

Title: Tractable Algorithms for Robust Model Estimation

Authors: O. Sapiro, E. Aik, F. Kahl, K. Astrom

Published: International Journal of Computer Vision 2015 [pdf]

Title: Two Efficient Solutions for Visual Odometry Using Directional Correspondence

Authors: O. Naroditsky, X.S. Zhou, J.A. Gallier, S.I. Roumeliotis, K. Daniilidis

Published: IEEE Trans. Pattern Anal. Mach. Intell. 34, 2012 [http]

Showing 1 to 100 of 105 entries

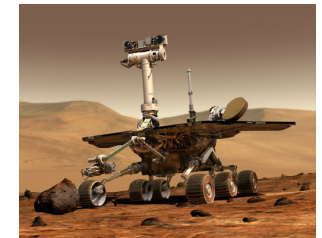
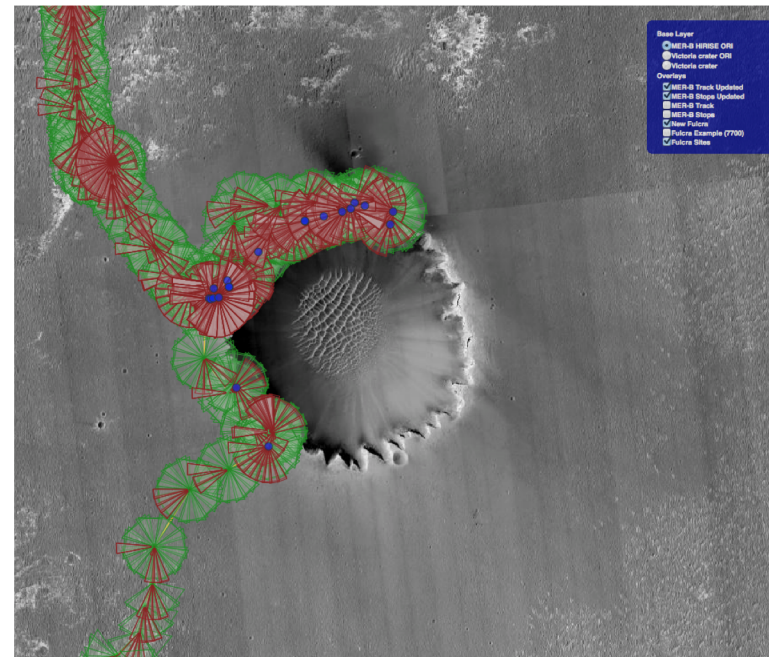
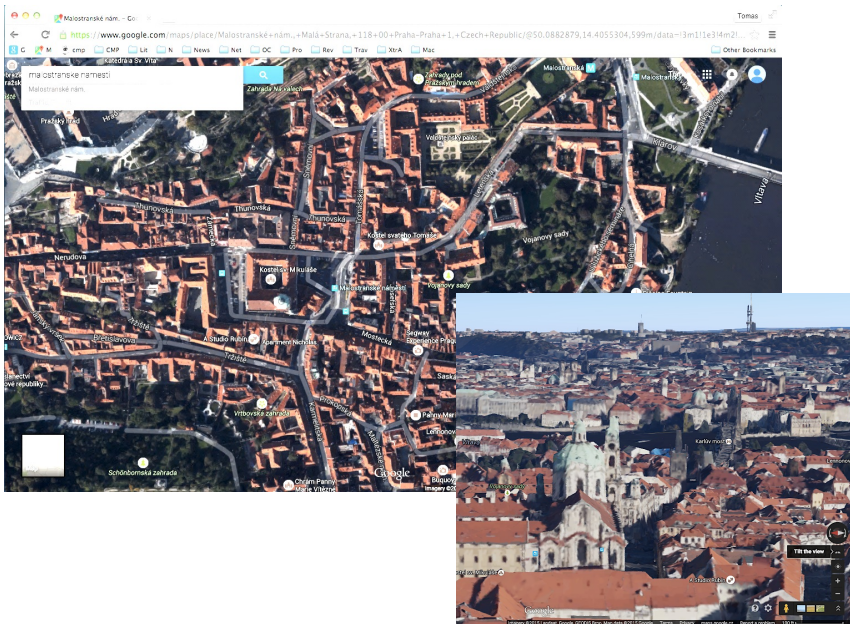
Previous 1 2 Next

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http://cmp.felk.cvut.cz/minimal/

pajdla@cvut.cz

Applications



Solving Minimal Problems

Minimal problem:

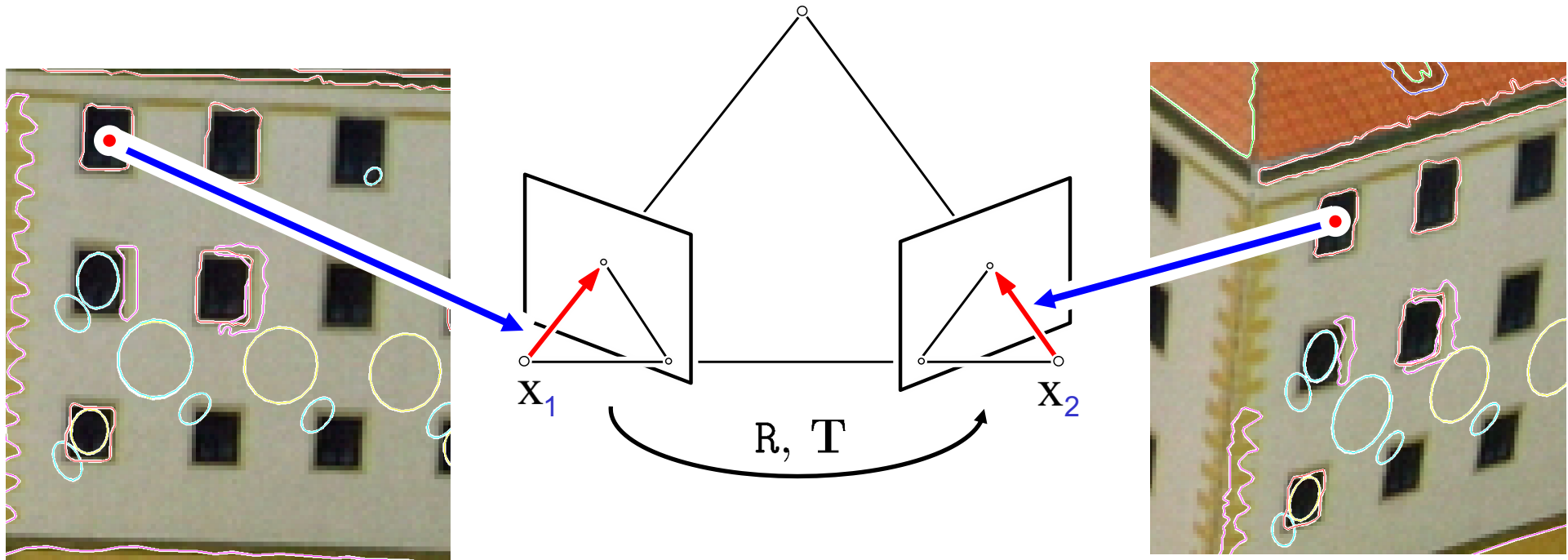
a problem that leads to solving a system of algebraic equations with a finite number of solutions.

1. Problem formulation \rightarrow algebraic equations
2. Solve algebraic equations

Easy? ... **NO**

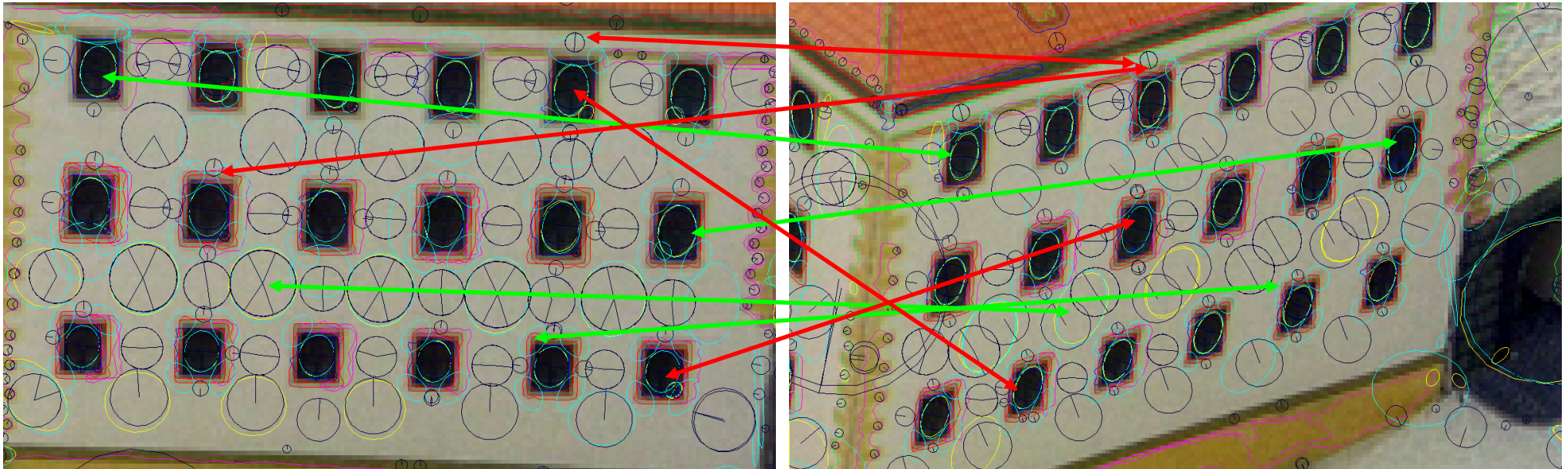
We have to be very fast in computer vision applications!

Why to be fast?



Solvers are used in combinatorial optimization

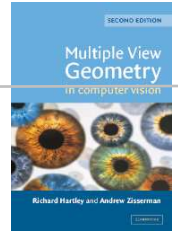
Image matching



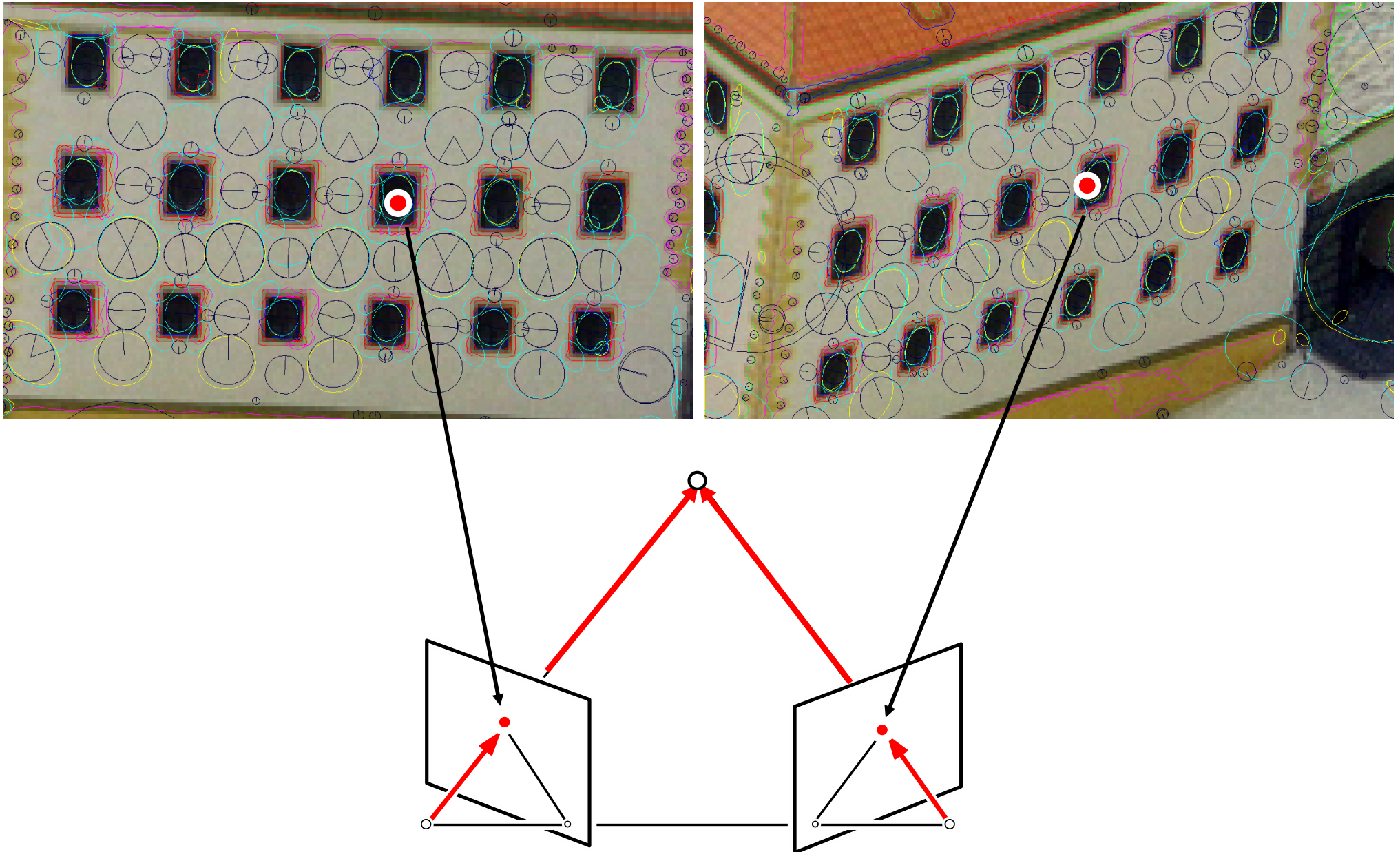
- Similar objects (circles, rectangles) ... tentative matches
- Some are correct, some are wrong
- Optimization task:

Find the largest (large) subset of tentative matches
consistent with a valid two view geometry

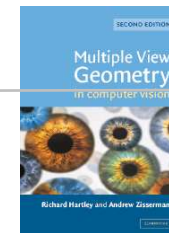
Valid two view geometry



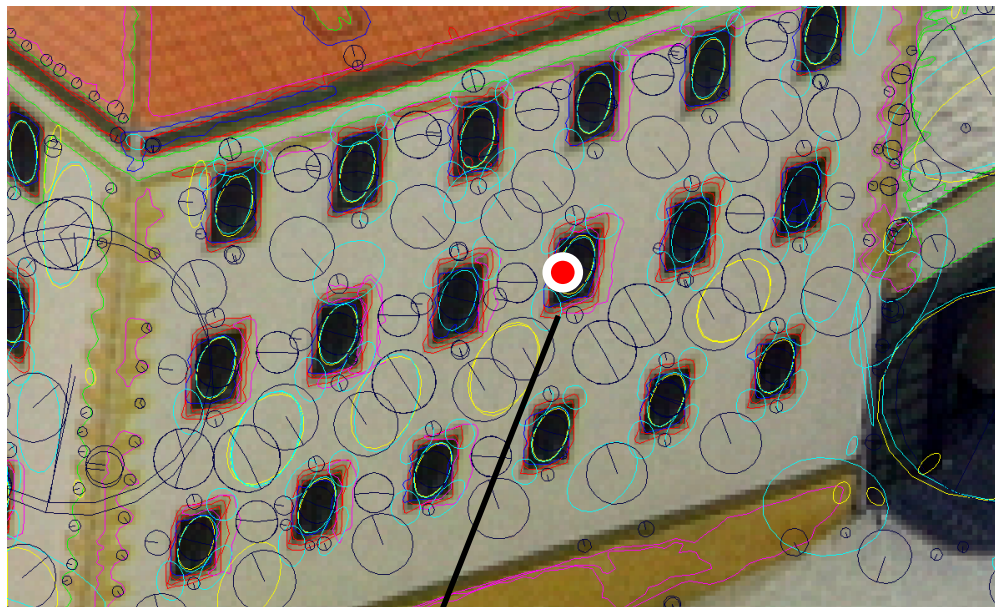
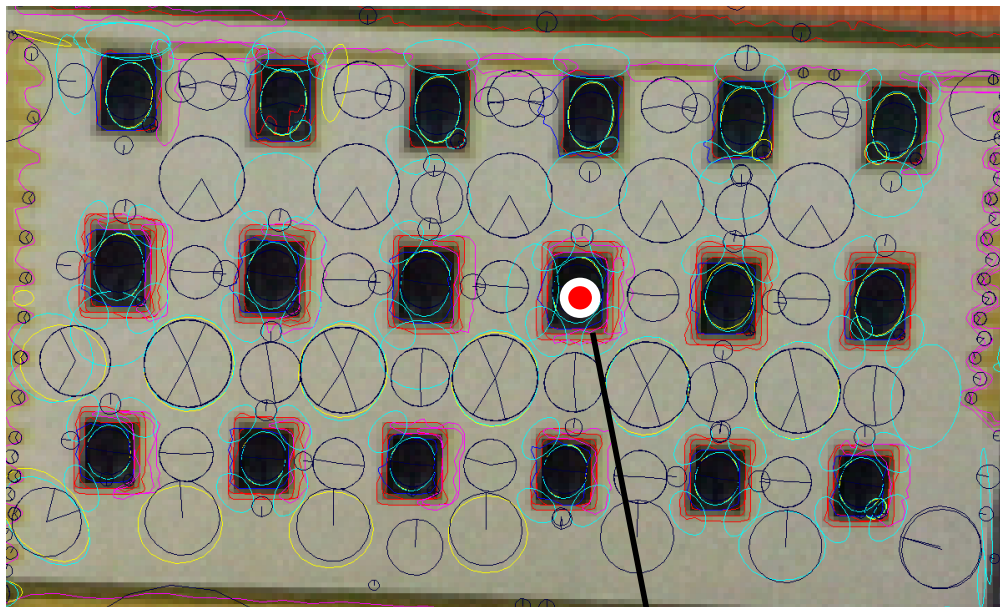
Epipolar constraint



Valid two view geometry

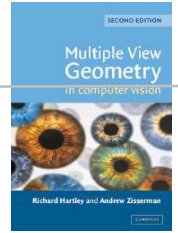


Epipolar constraint

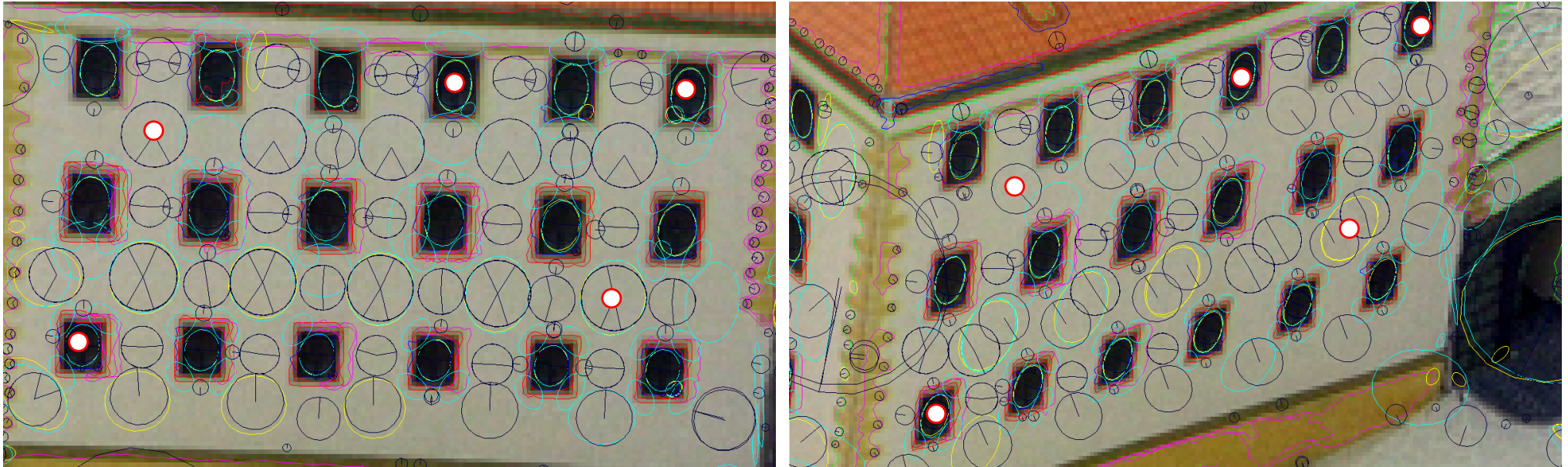


$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

Valid two view geometry



Epipolar constraint

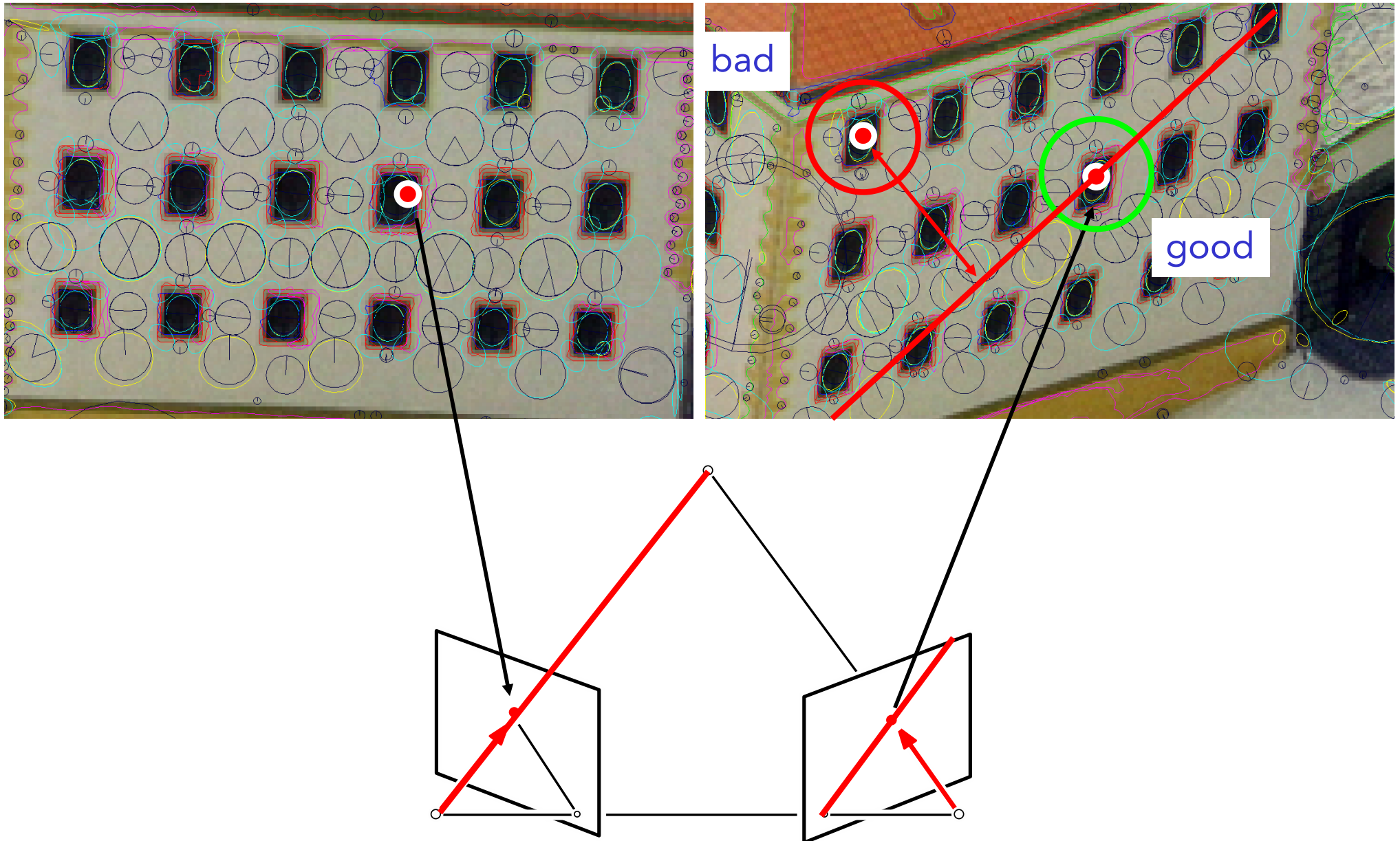


- F can be computed from 5 matches
- The best F is consistent with the largest subset

$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

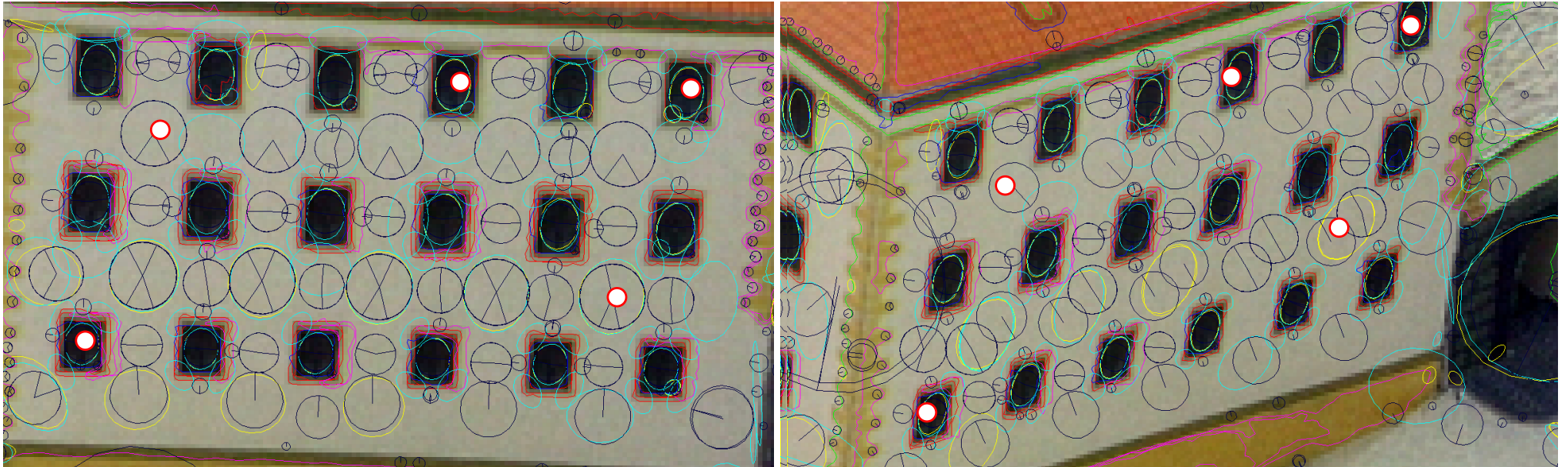
Consistent two view geometry

Matching constraint



Optimization scheme = RANSAC

Enumerating all subsets replaced by checking only some of them



RANdom SAMpling Consensus

1. Generate random 5-tuples of matches
2. Compute F by solving $\mathbf{x}_2^T F \mathbf{x}_1 = 0$
3. Count the number of good matches

Return the largest set of good matches

Why to be fast?

- Many samples needed to be sure to find a good sample!

To find a gross-error-free sample with 95% probability we have to try at least the following number of samples:

Gross error contamination ratio [%]

Sample size	15%	20%	30%	40%	50%	70%	
	2	132	73	32	17	10	4
	4	5916	1871	368	116	46	11
	7	$1.75 \cdot 10^6$	$2.34 \cdot 10^5$	$1.37 \cdot 10^4$	1827	382	35
	8	$1.17 \cdot 10^7$	$1.17 \cdot 10^6$	$4.57 \cdot 10^4$	4570	765	50
	12	$2.31 \cdot 10^{10}$	$7.31 \cdot 10^8$	$5.64 \cdot 10^6$	$1.79 \cdot 10^5$	$1.23 \cdot 10^4$	215
	18	$2.08 \cdot 10^{15}$	$1.14 \cdot 10^{13}$	$7.73 \cdot 10^9$	$4.36 \cdot 10^7$	$7.85 \cdot 10^5$	1838
	30	∞	∞	$1.35 \cdot 10^{16}$	$2.60 \cdot 10^{12}$	$3.22 \cdot 10^9$	$1.33 \cdot 10^5$
	40	∞	∞	∞	$2.70 \cdot 10^{16}$	$3.29 \cdot 10^{12}$	$4.71 \cdot 10^6$

Solving time: micro-mili seconds

How to be fast?

How to be fast?

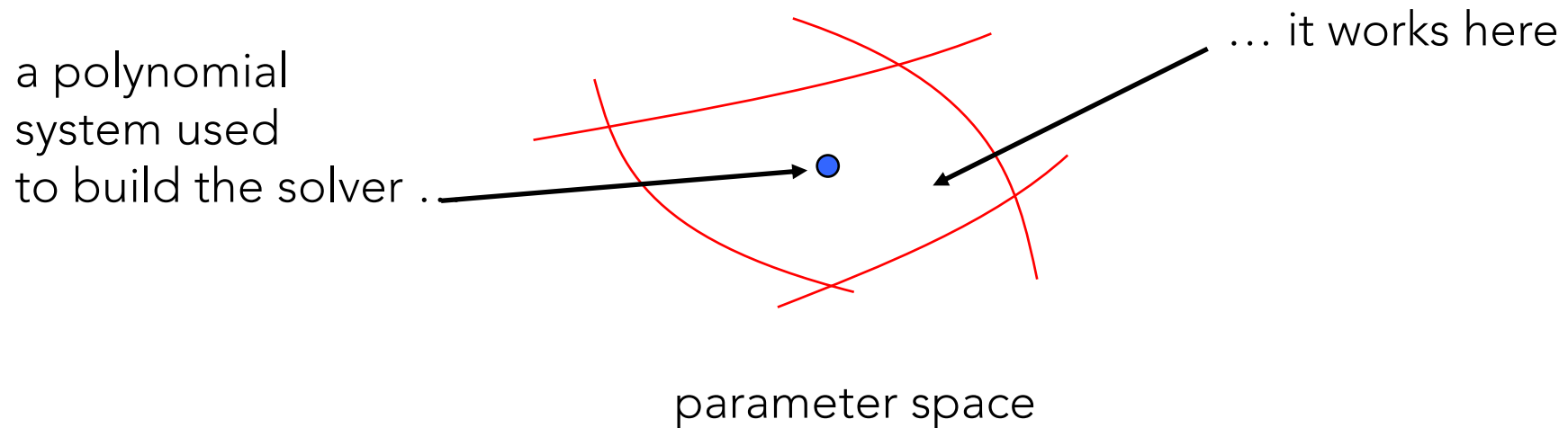
1. Specialized solving methods
2. Assume generic data
3. Use tricks, optimize, hard code, ...

Many problems are generic

Solvers do not (much) differ from one problem to another.

→ Solver is made out by solving a single concrete system and then used on other systems

→ This works around generic solutions



Strategy of fast solving

Offline phase (may be slow)

1. **Fabricate a concrete generic example** of a polynomial system (generating 0-dim radical ideal I)
2. **Analyze the system** by a generic method (Macaulay2, FGb, ...) to get the degree, (standard monomial) basis in R/I , ...
3. **Create an elimination template** for constructing a multiplication matrix M_f of multiplication by a suitable polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ (an unknown) in a finite-dimensional factor ring $A = \mathbb{C}[x_1, \dots, x_n]/I$.
4. **Implement efficiently** in floating points, optimize, test, ... (vary ordering, basis selection, ...)

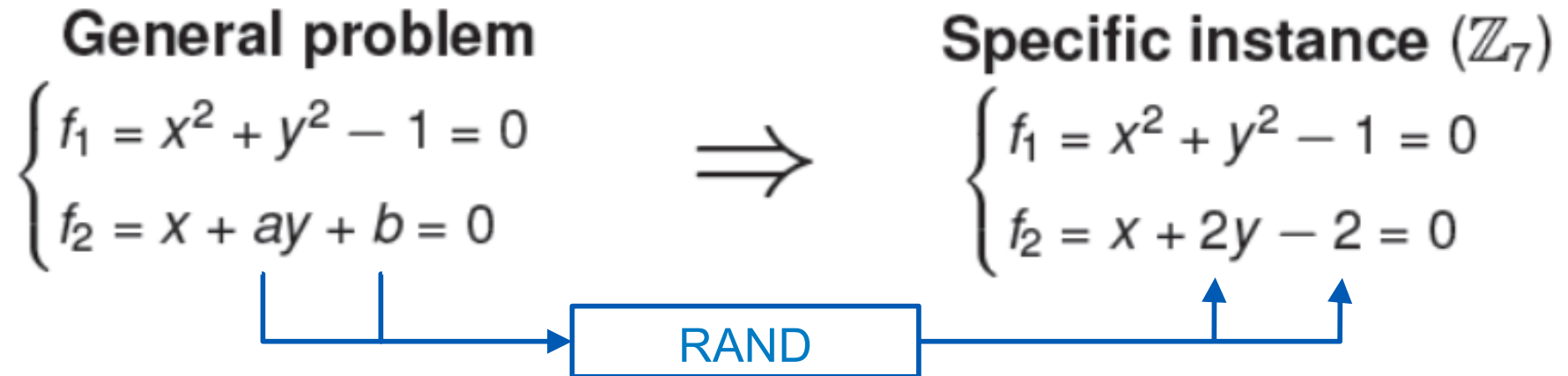
Strategy of fast solving

Online (**must be fast**)

1. **Fill the elimination template** to get matrix M_f .
2. **Solve numerically** by finding eigenvectors of M_f .
(or get a univariate poly and use real root bracketing)

Offline: Fabricate a concrete generic example

1. An easy example

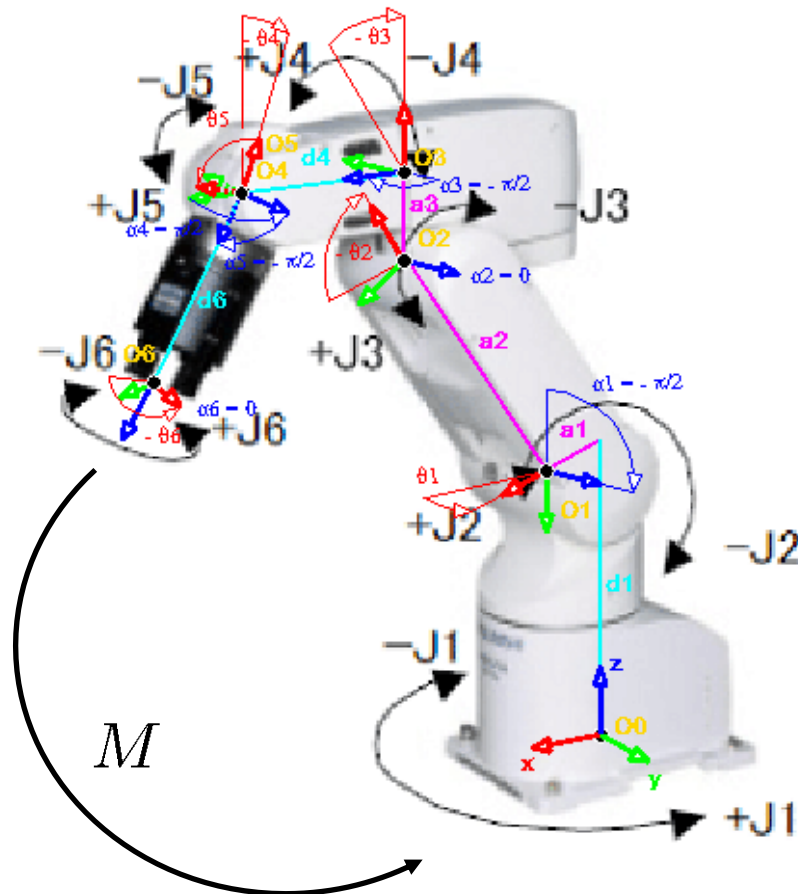


Offline: Fabricate a concrete generic example

2. A (not difficult) example (Inverse Kinematic Task in robotics)

Given M find c_i, s_i (sin & cos of controlled angles)

$$M = M_1^0(c_1, s_1) M_2^1(c_2, s_2) M_3^2(c_3, s_3) M_4^3(c_4, s_4) M_5^4(c_5, s_5) M_6^5(c_6, s_6)$$



$$M_i^{i-1} = \begin{bmatrix} c_i & -s_i p_i & s_i q_i & a_i c_i \\ s_i & c_i p_i & -c_i q_i & a_i s_i \\ 0 & q_i & p_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} c_1^2 + s_1^2 &= 1 & c_4^2 + s_4^2 &= 1 \\ c_2^2 + s_2^2 &= 1 & c_5^2 + s_5^2 &= 1 \\ c_3^2 + s_3^2 &= 1 & c_6^2 + s_6^2 &= 1 \end{aligned}$$

- M must contain a rotation matrix to get a consistent system.
- A rational rotation must be constructed (no difficult)

Offline: Fabricate a concrete generic example

3. Hard cases exist too

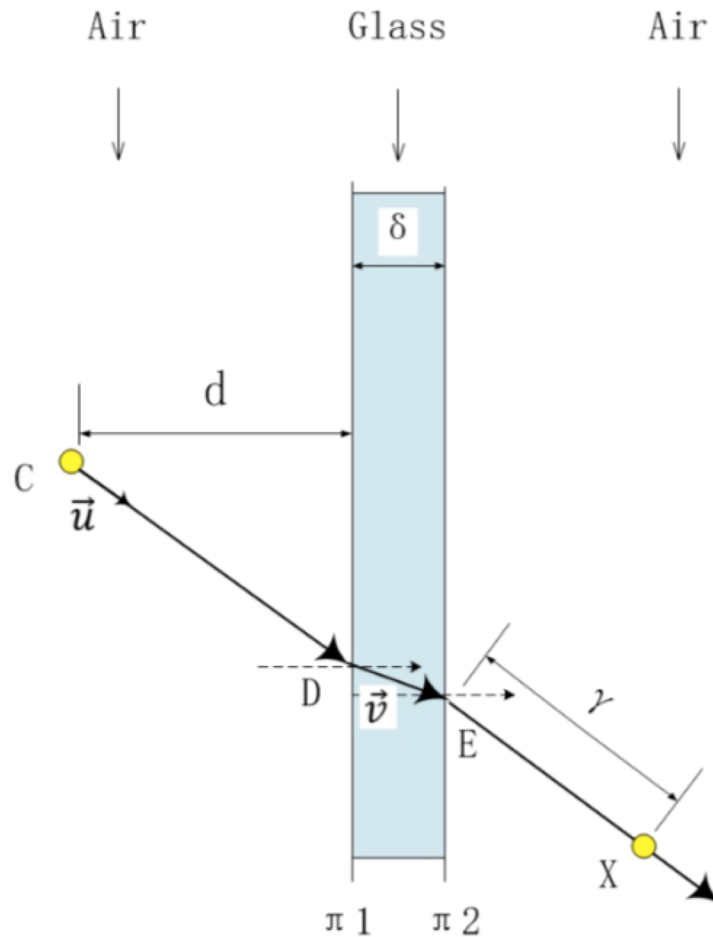


Figure 3.1: Model of the planar glass and projection geometry

A more general parametric systems:

Do we need to find a rational point on a variety?

- Hard?
- When possible/impossible?
- Other options (numerical) if impossible?

Offline: Analyze the system

Specific instance (\mathbb{Z}_7)

$$\begin{cases} f_1 = x^2 + y^2 - 1 = 0 \\ f_2 = x + 2y - 2 = 0 \end{cases}$$

```
Macaulay2, version 1.12
```

```
-- Ring
```

```
i1 : R = ZZ/7[x,y]
```

```
-- An instance
```

```
i2 : F = matrix({{x^2+y^2-1},{x+2*y-2}})
```

```
i3 : I = ideal(F)
```

```
-- The dimension and the degree of V(I)
```

```
i4 : dim(I), degree(I)
```

```
o4 = (0, 2)
```

```
-- The standard monomial basis of A = R/I
```

```
i5 : A = R/I
```

```
i6 : B = basis(A)
```

```
o6 = | y 1 |
```


Offline: Create an elimination template

$$y \ B = \boxed{M} \ B + \overbrace{\left[\begin{array}{cc} T & F \end{array} \right]}^{\in I} \left(\begin{array}{c} x^2 + y^2 - 1 \\ x + 2x + 2 \end{array} \right)$$

$$y \begin{pmatrix} y \\ 1 \end{pmatrix} = \boxed{\begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix}} \begin{pmatrix} y \\ 1 \end{pmatrix} + \left[\begin{array}{cc} 3 & -3x - y + 1 \\ 0 & 0 \end{array} \right] \begin{pmatrix} x^2 + y^2 - 1 \\ x + 2x + 2 \end{pmatrix}$$

x^2	xy	x	y^2	y	1
1			1		-1
1	a	b			
	1		a	b	
		1		a	b

$$\begin{array}{lcl} f_1 & \leftarrow 1 & \text{---} x^2 + y^2 - 1 \\ x f_2 & \leftarrow x & \text{---} \\ y f_2 & \leftarrow y & \text{---} x + a x + b \\ f_2 & \leftarrow 1 & \text{---} \end{array}$$

x^2	xy	x	y^2	y	1
1				•	•
	1			•	•
		1		•	•
			1	$-m_{11}$	$-m_{12}$

Gauss-Jordan Elimination

Offline: Create an elimination template

```
Macaulay2
```

```
-- Reduce y multiple of B by I
```

```
i9 : r = (y*B) % I
```

```
o9 = | 3y-2 y |
```

```
-- Multiplication matrix of y in R/I w.r.t. B
```

```
i10 : M = transpose((coefficients(r,Monomials=>B))_1)
```

```
o10 = {-1} | 3 -2 |
```

```
      {0} | 1 0 |
```

```
-- Create the template for constructing M from F
```

```
-- y*B = M*B + T*F
```

```
i12 : T = transpose(transpose(y*B-M*B) // gens(I))
```

```
o12 = {-1} | 3 -3x-y+1 |
```

```
      {0} | 0 0 |
```

```
-- Extract monomials from columns of T
```

```
i14 : m = apply(m,x->apply(x,x->( ... (T) ... ))
```

```
o14 = {{1}, {1, y, x}}
```

Offline: Create an elimination template

```
-- Create the template for constructing M from F
--  $y*B = M*B + T*F$ 
i12 : T = transpose(transpose(y*B-M*B) // gens(I))
o12 = {-1} | 3 -3x-y+1 |
      {0} | 0 0 |
```

// ... does the magic

- Traces Groebner basis construction
- Reduces the template by the Groebner basis of the Syzygy module
- Often produces very efficient templates (no unnecessary rows) but not always
- When can be “the optimal template” defined and found? (F5?)
- Other parameters important ... e.g. basis in R/I selection

Online: Fill the template, get M , eigenvectors

New coefficients a, b

$$\begin{aligned} x^2 + y^2 - 1 \\ x + ax + b \end{aligned}$$



x^2	xy	x	y^2	y	1
1			1		-1
1	a	b			
	1		a	b	
		1		a	b

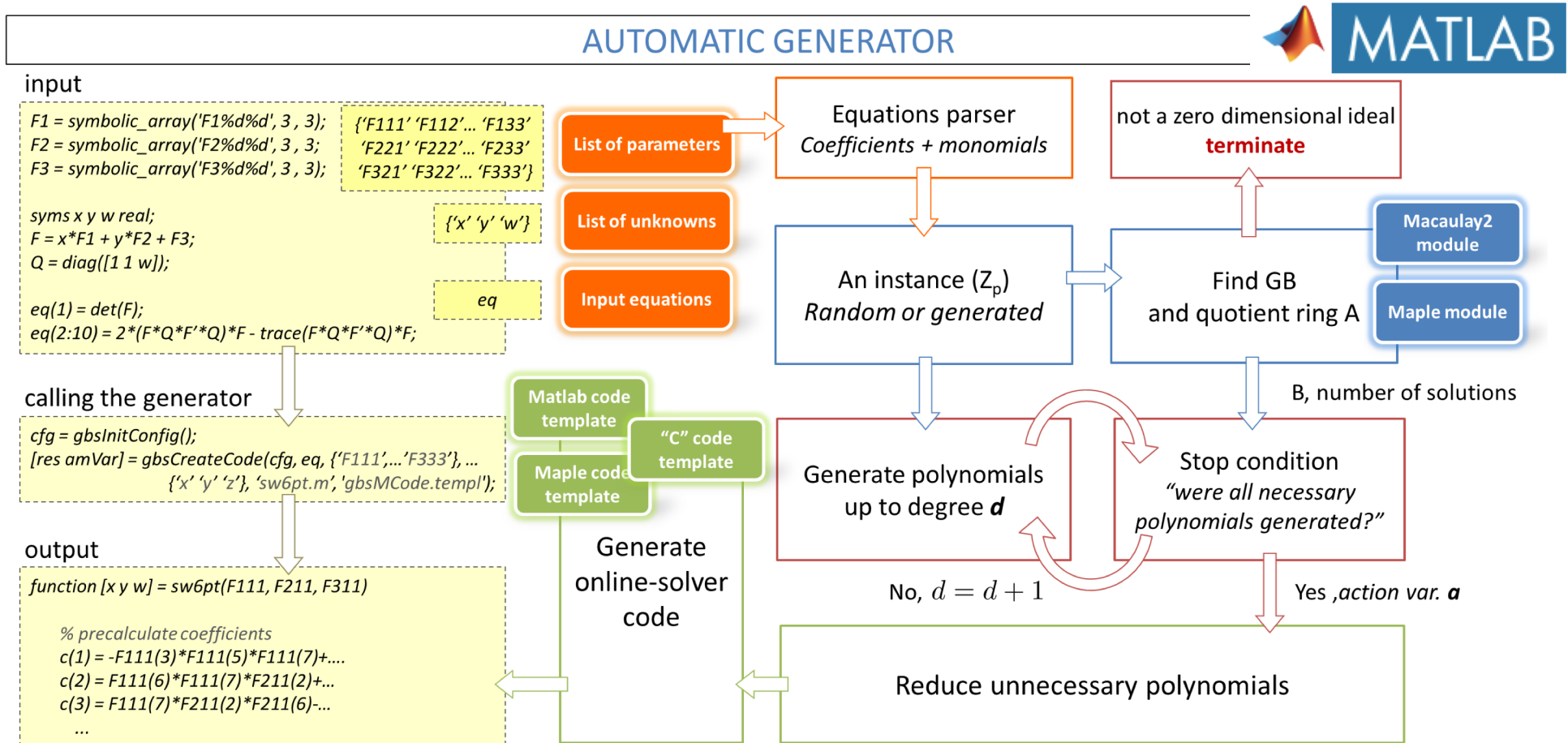
Gauss-Jordan
Elimination

x^2	xy	x	y^2	y	1
1				•	•
	1			•	•
		1		•	•
			1	$-m_{11}$	$-m_{12}$



Compute solutions $v \simeq \begin{pmatrix} y \\ 1 \end{pmatrix}$ by $\lambda v = \begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix} v$

Automatic generator of “minimal solvers”



- Z Kukelova, M Bujnak, T Pajdla.
[Automatic Generator of Minimal Problem Solvers](#). ECCV 2008.
- V Larsson, K Astrom, M Oskarsson.
[Efficient Solvers for Minimal Problems by Syzygy-Based Reduction](#). CVPR 2017.
- V Larsson, M Oskarsson, K Astrom, A Wallis, Z Kukelova, T Pajdla.
[Beyond Grobner Bases: Basis Selection for Minimal Solvers](#). CVPR 2018

Template construction optimization

- Criteria for the best template have not (yet) been clearly defined but
- Efficient templates are small and numerically robust
- R/I basis selection is important (for the strategy described)

Basis selection in $A = R/I$

- How to choose a good basis?
- Experiments with monomial orderings + more ...

Basis selection in $A = R/I$

- Only a finitely many different Groebner bases (Groebner fan)

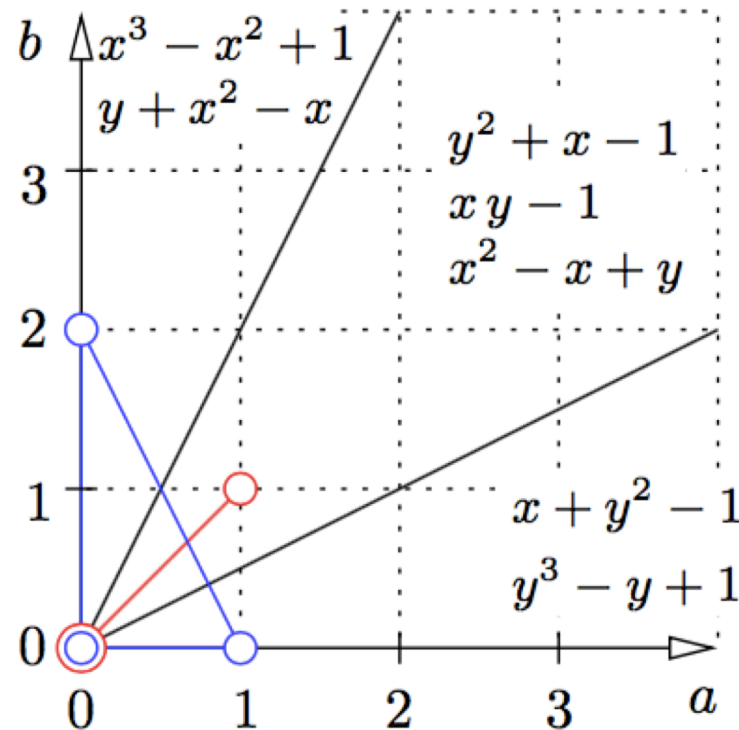


Figure 1. The Gröbner fan of the ideal $I = \langle x + y^2 - 1, xy - 1 \rangle$ consists of three two-dimensional cones. For each cone, there is exactly one reduced Gröbner basis of I . All monomial orderings generated by all weight vectors from one cone give the same reduced Gröbner basis of I . Hence, there are exactly three different reduced Gröbner bases for I over all possible different monomial orderings.

Basis selection in $A = R/I$

- Generate standard monomial bases for all GB in the GB fan.
- Test them and choose **the best** (the smallest and stable) basis.
- Go beyond: Use a heuristic to sample other **even better** bases.

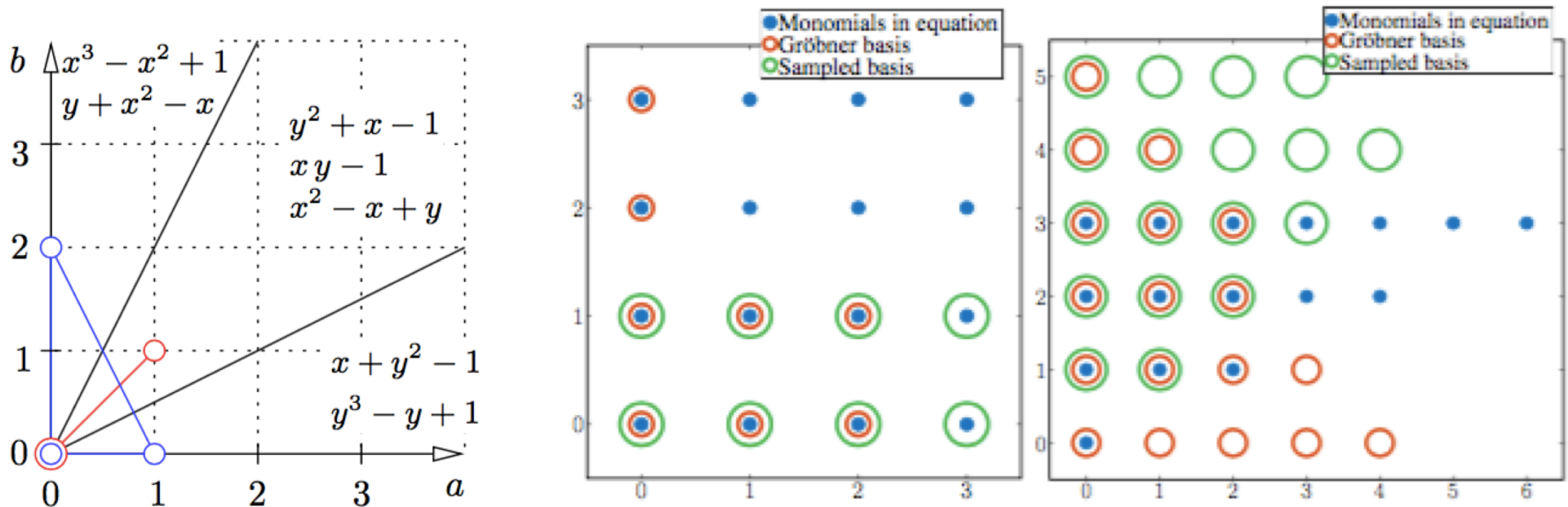


Figure 3. The figure shows the basis monomials for two example problems, namely 8pt rel. pose $F+\lambda$ (left) and 3pt image stitching $f\lambda+R+f\lambda$ (right). Both these problems have two variables, and for both these problems the proposed basis sampling scheme gives significantly smaller template compared to the Gröbner basis vari-

Basis selection in $A = R/I$

- There are dramatic differences

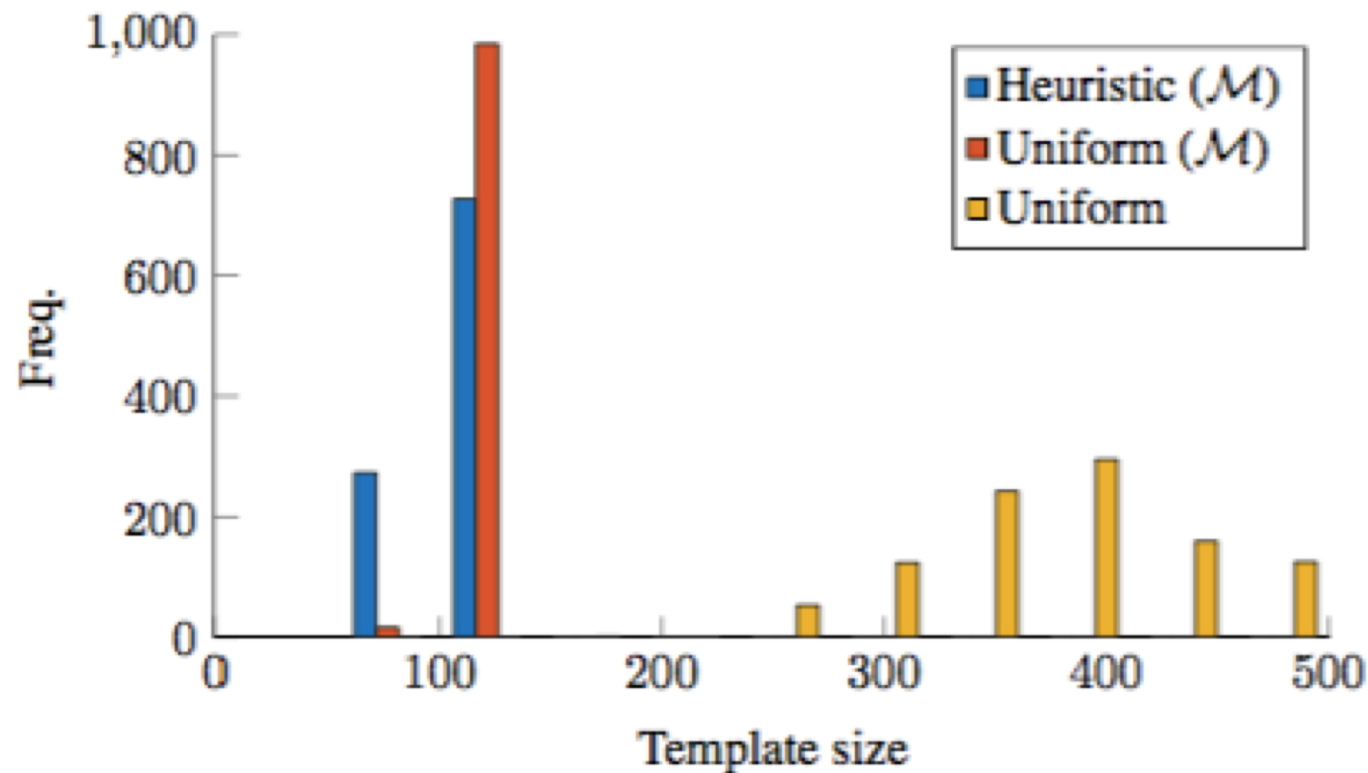


Figure 4. Template size (rows) for 1,000 randomly sampled bases for the P4Pfr formulation from Bujnak *et al.* [7].

Basis selection in $A = R/I$

- Many solvers improved

Problem	Author	Original	[29]	GFan+ [29]	(#GB)	Heuristic+[29]
Rel. pose $F+\lambda$ 8pt	Kuang <i>et al.</i> [25]	12×24	11×20	11×20	(10)	7×16
Rel. pose $E+f$ 6pt	Bujnak <i>et al.</i> [6]	21×30	21×30	11×20	(66)	11×20
Rel. pose $f+E+f$ 6pt	Kukelova <i>et al.</i> [26]	31×46	31×50	31×50	(218)	21×40
Rel. pose $E+\lambda$ 6pt	Kuang <i>et al.</i> [25]	48×70	34×60	34×60	(846)	14×40
Stitching $f\lambda+R+f\lambda$ 3pt	Naroditsky <i>et al.</i> [35]	54×77	48×66	48×66	(26)	18×36
Abs. Pose P4Pfr	Bujnak <i>et al.</i> [7]	136×152	140×156	54×70	(1745)	54×70
Rel. pose $\lambda+E+\lambda$ 6pt	Kukelova <i>et al.</i> [26]	238×290	149×205	-	?	53×115
Rel. pose $\lambda_1+F+\lambda_2$ 9pt	Kukelova <i>et al.</i> [26]	179×203	165×200	84×117	(6896)	84×117
Rel. pose $E+f\lambda$ 7pt	Kuang <i>et al.</i> [25]	200×231	181×200	69×90	(3190)	69×90
Rel. pose $E+f\lambda$ 7pt (elim. λ)	-	-	52×71	37×56	(332)	24×43
Rel. pose $E+f\lambda$ 7pt (elim. $f\lambda$)	Kukelova <i>et al.</i> [27]	51×70	51×70	51×70	(3416)	51×70
Abs. pose quivers	Kuang <i>et al.</i> [22]	372×386	216×258	-	?	81×119
Rel. pose E angle+4pt	Li <i>et al.</i> [32]	270×290	266×329	-	?	183×249
Abs. pose refractive P5P	Haner <i>et al.</i> [17]	280×399	240×324	157×246	(8659)	240×324

Table 1. Size of the elimination templates for some minimal problems. For the relative pose problems unknown radial distortion is denoted with λ and unknown focal length with f , and the position describes which camera it refers to. The table shows the original template size from the author, the template size found using the method from [29] (GRevLex basis), the template size from doing an exhaustive search over Gröbner bases (Section 2.2) and the random sampling approach (Section 3.1). Missing entries are when the Gröbner fan computation took longer than 12 hours.

Minimal Problems in Computer Vision

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