Minimal Problems in Computer Vision

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in collaboration with

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Joe Kileel, Bernd Sturmfels

3D Reconstruction from Photographs

Capturing Reality (capturingreality.com)

Camera & structure computation essential …

3D reconstruction

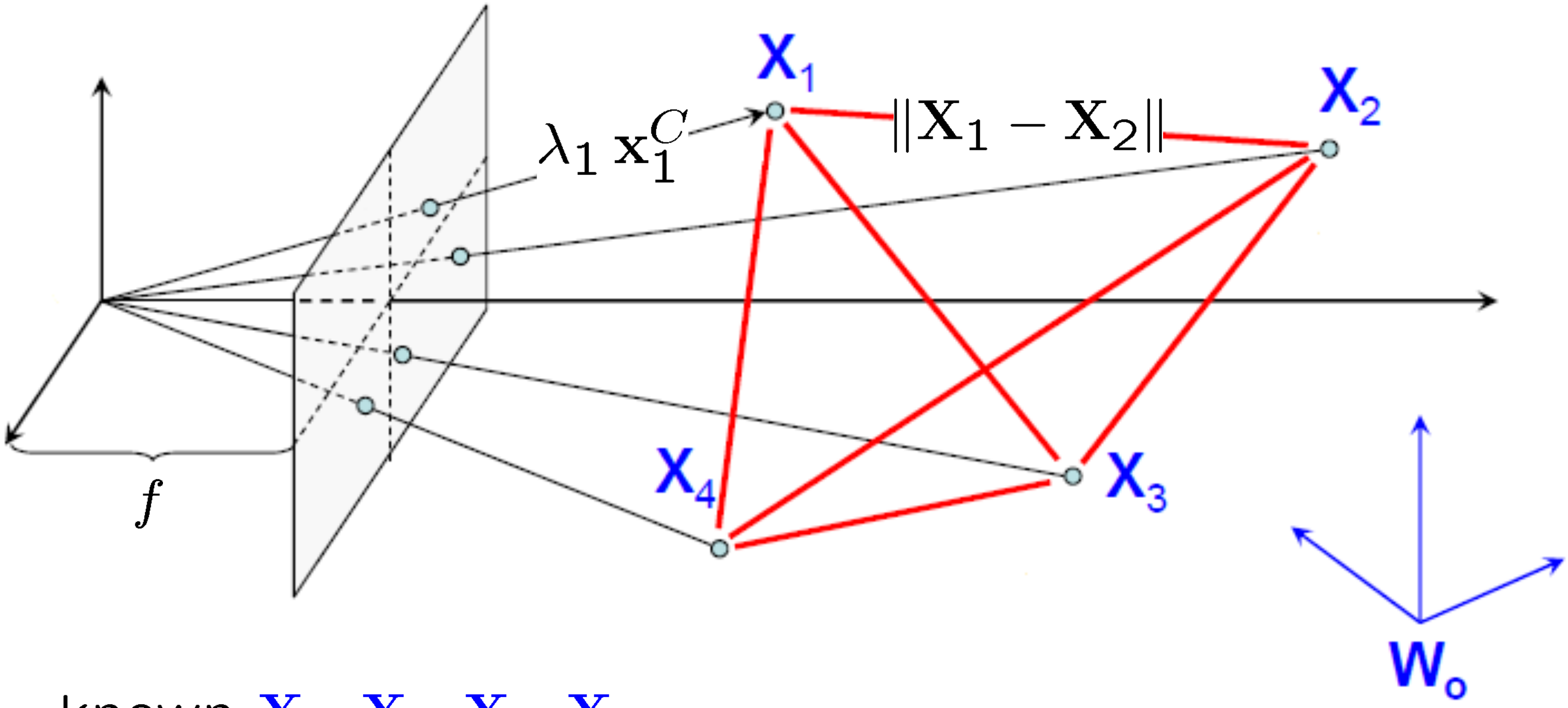
⬇

Solving “Minimal Problems”

by

Algebraic Geometry

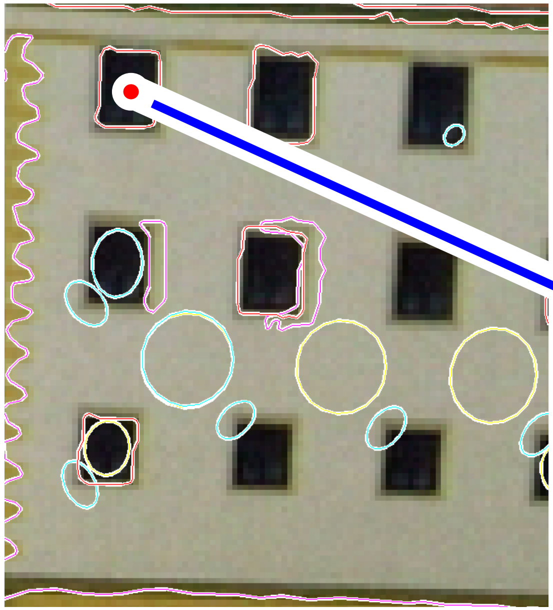
Minimal problem: Absolute Camera Orientation

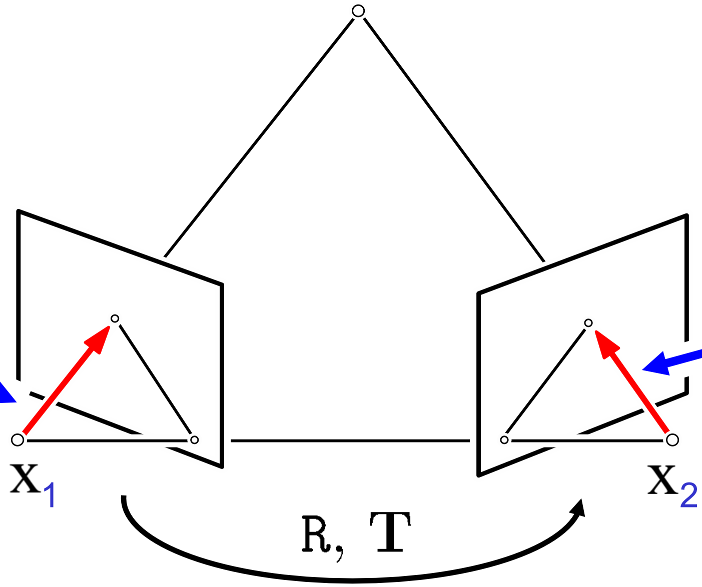


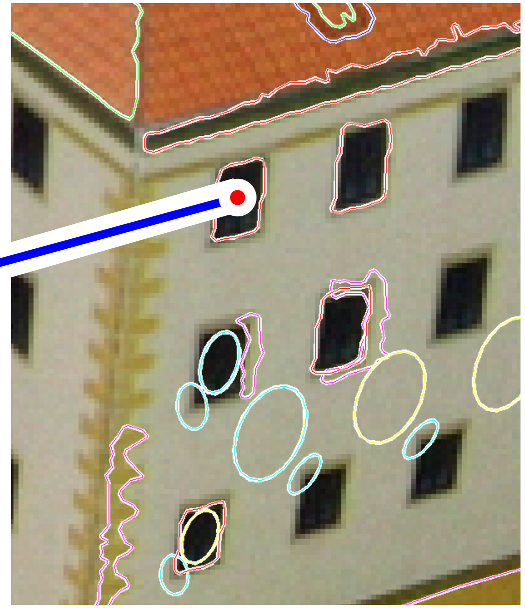
- known $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4$
- unknown $\mathtt{R}, \mathbf{T}, f$

$$\underbrace{\frac{\|\mathbf{X}_i - \mathbf{X}_j\|^2}{\|\mathbf{X}_k - \mathbf{X}_l\|^2} = \frac{\|\lambda_i\, \mathbf{x}_i^C - \lambda_j\, \mathbf{x}_j^C\|^2}{\|\lambda_k\, \mathbf{x}_k^C - \lambda_l\, \mathbf{x}_l^C\|^2}}_{\text{Algebraic equations}} \xrightarrow{\text{solve}} \lambda_i \longrightarrow \mathtt{R}, \mathbf{T}, f$$

Minimal problem: Relative Camera Orientation







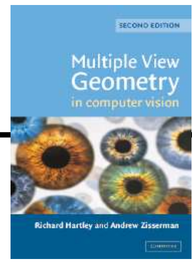
$$\mathtt{x}_2^\top \mathtt{F}\,\mathtt{x}_1 = 0$$

$$\det \mathtt{F} = 0$$

$$2\,\mathtt{F}\mathtt{F}^\top\mathtt{F} - \mathrm{trace}(\mathtt{F}\mathtt{F}^\top)\mathtt{F} = 0$$

Algebraic equations
from 5 correspondences

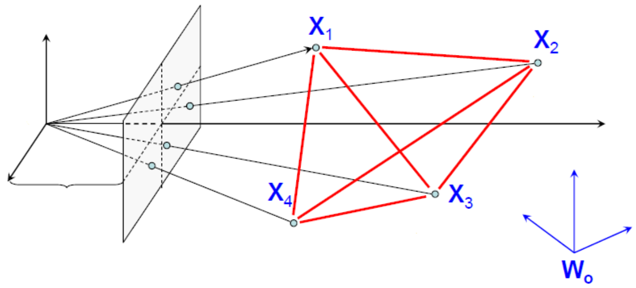
$\xrightarrow{\text{solve}}$ $\mathtt{F}$ —



$\longrightarrow \mathtt{R}, \dfrac{\mathbf{T}}{\|\mathbf{T}\|}$

Long history of Minimal Problems

Absolute Camera Orientation



1841 J. A. Grunert

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1981 M. A. Fischler and R. C. Bolles

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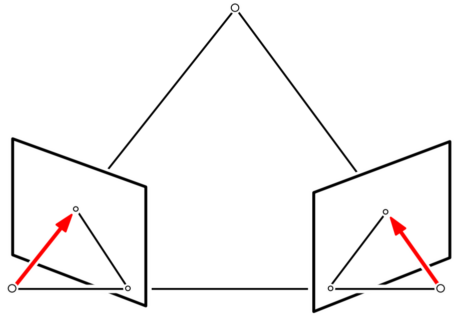
2013 Z. Kukelova et al.

Relative Camera Orientation



1913 E. Kruppa

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1981 H. Longuet-Higgins

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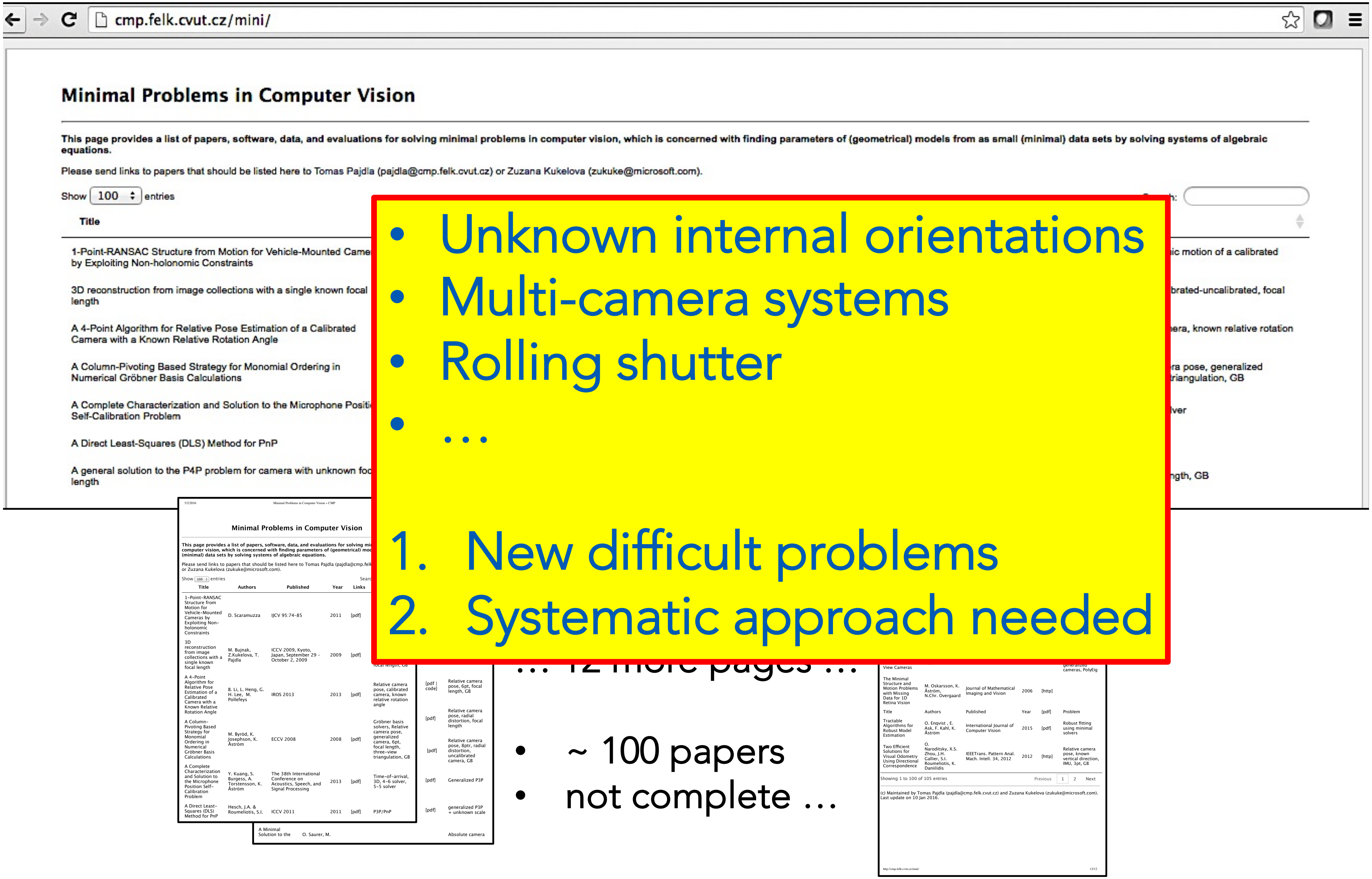
2004 D. Nister

- many papers
- many applications

Minimal problems list (cmp.felk.cvut.cz/minimal)



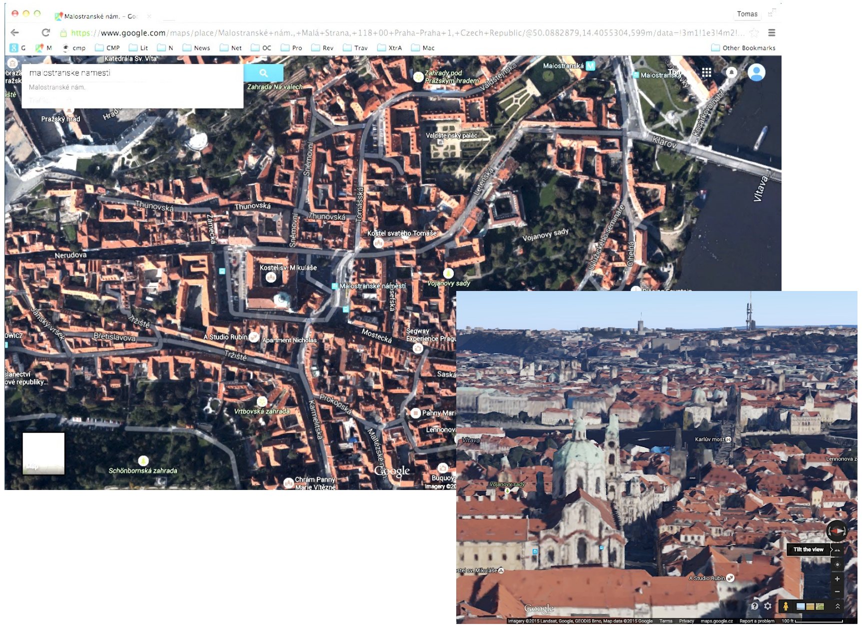
- Unknown internal orientations
- Multi-camera systems
- Rolling shutter
- …

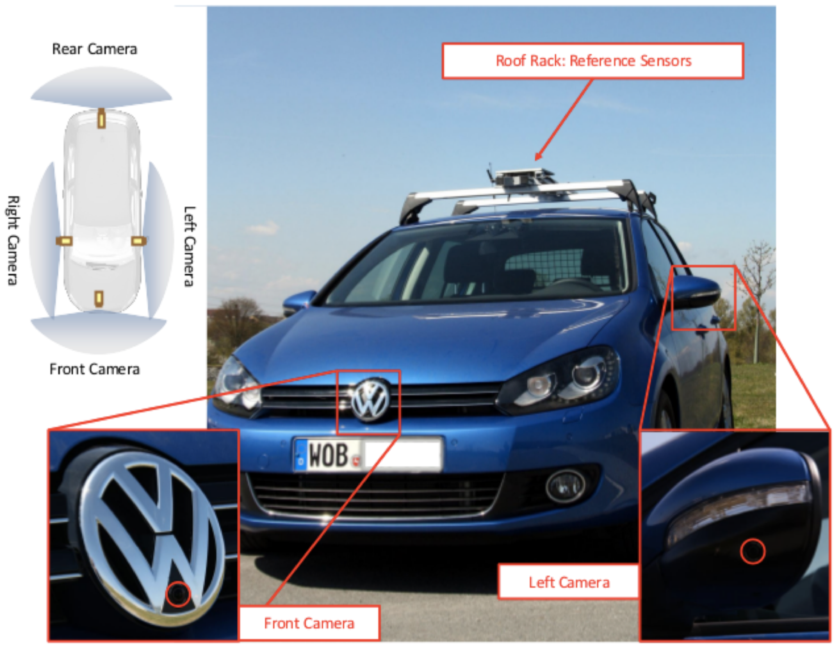
1. New difficult problems
2. Systematic approach needed

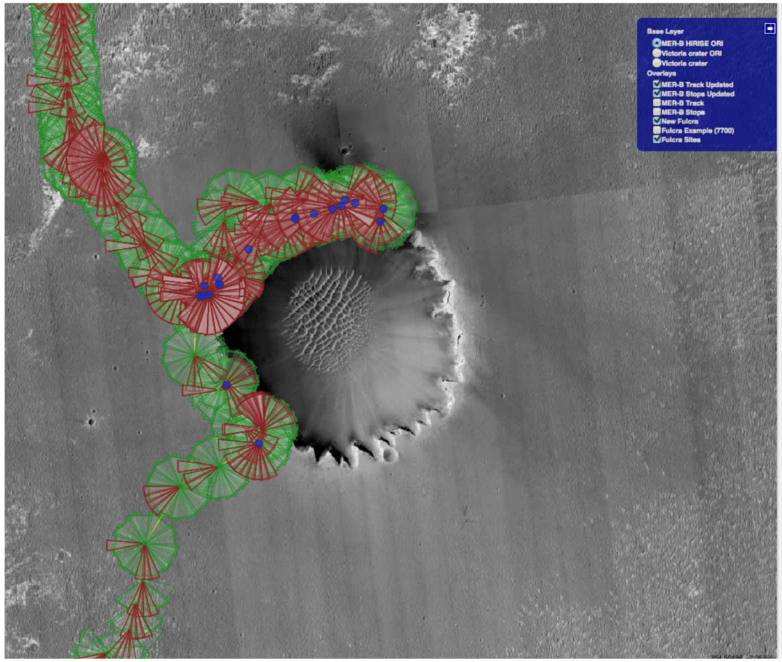
… 12 more pages …

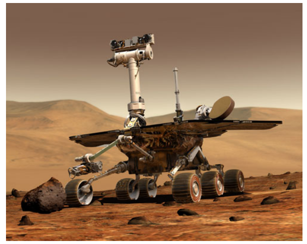
- ~ 100 papers
- not complete …

Applications









Solving Minimal Problems

Minimal problem:

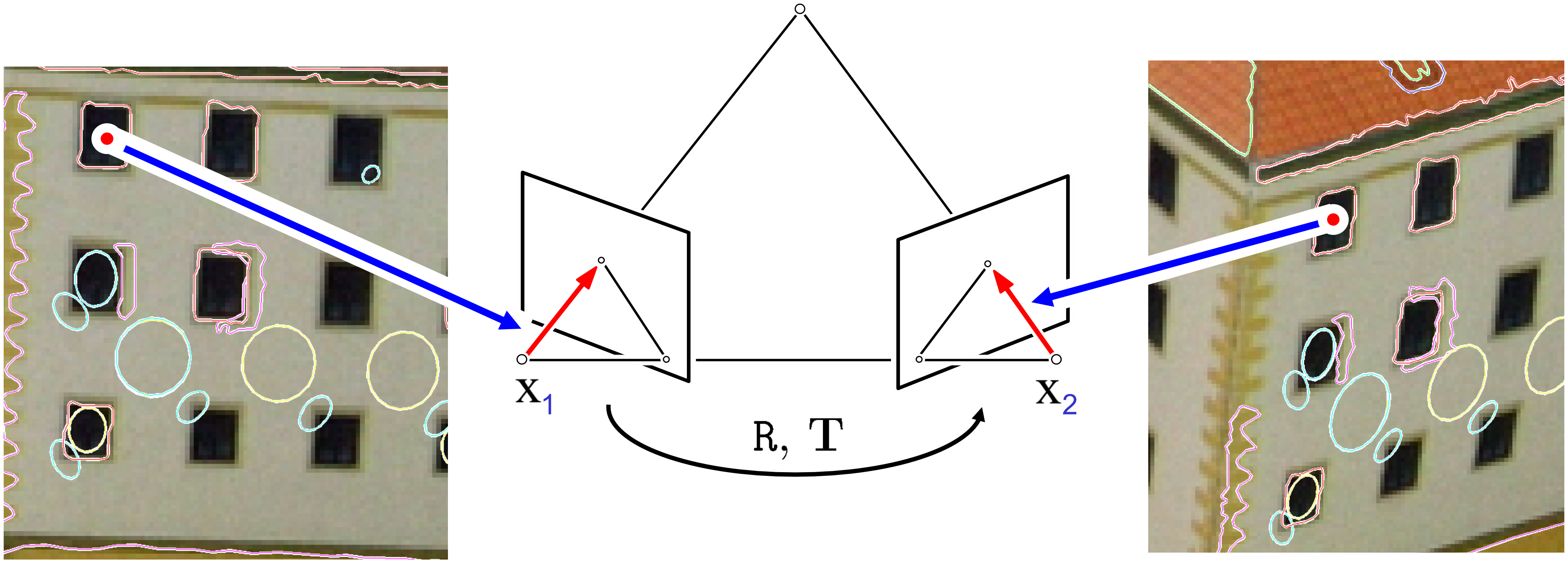
a problem that leads to solving a system of algebraic equations with a finite number of solutions.

1. Problem formulation $\rightarrow$ algebraic equations
2. Solve algebraic equations

Easy? … NO

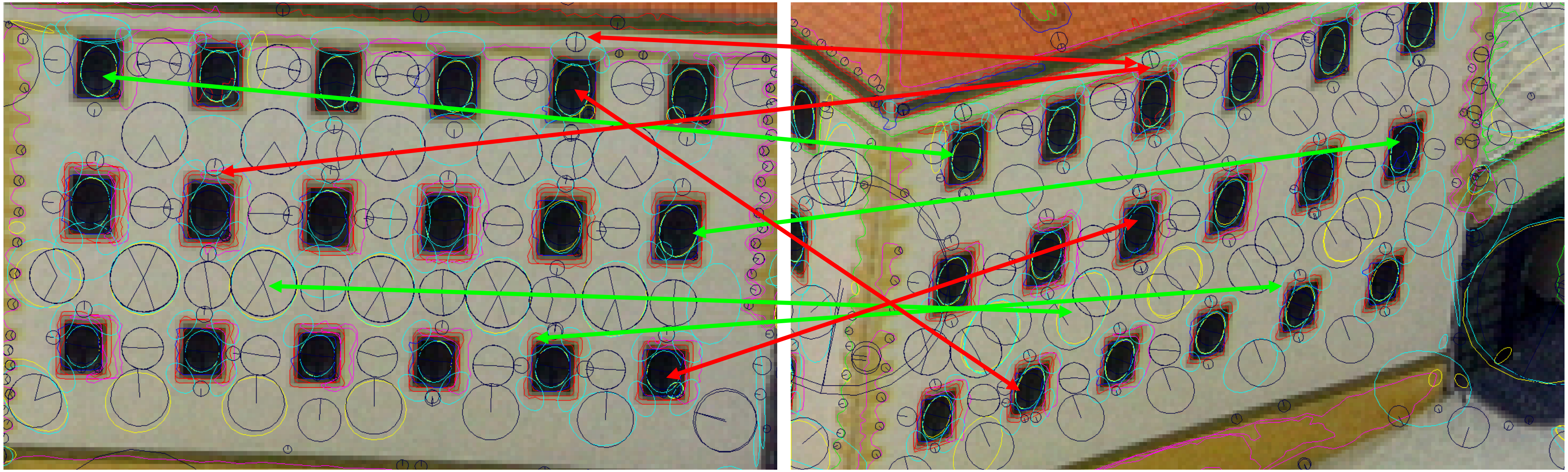
We have to be very fast in computer vision applications!

Why to be fast?



Solvers are used in combinatorial optimization

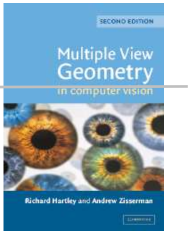
Image matching



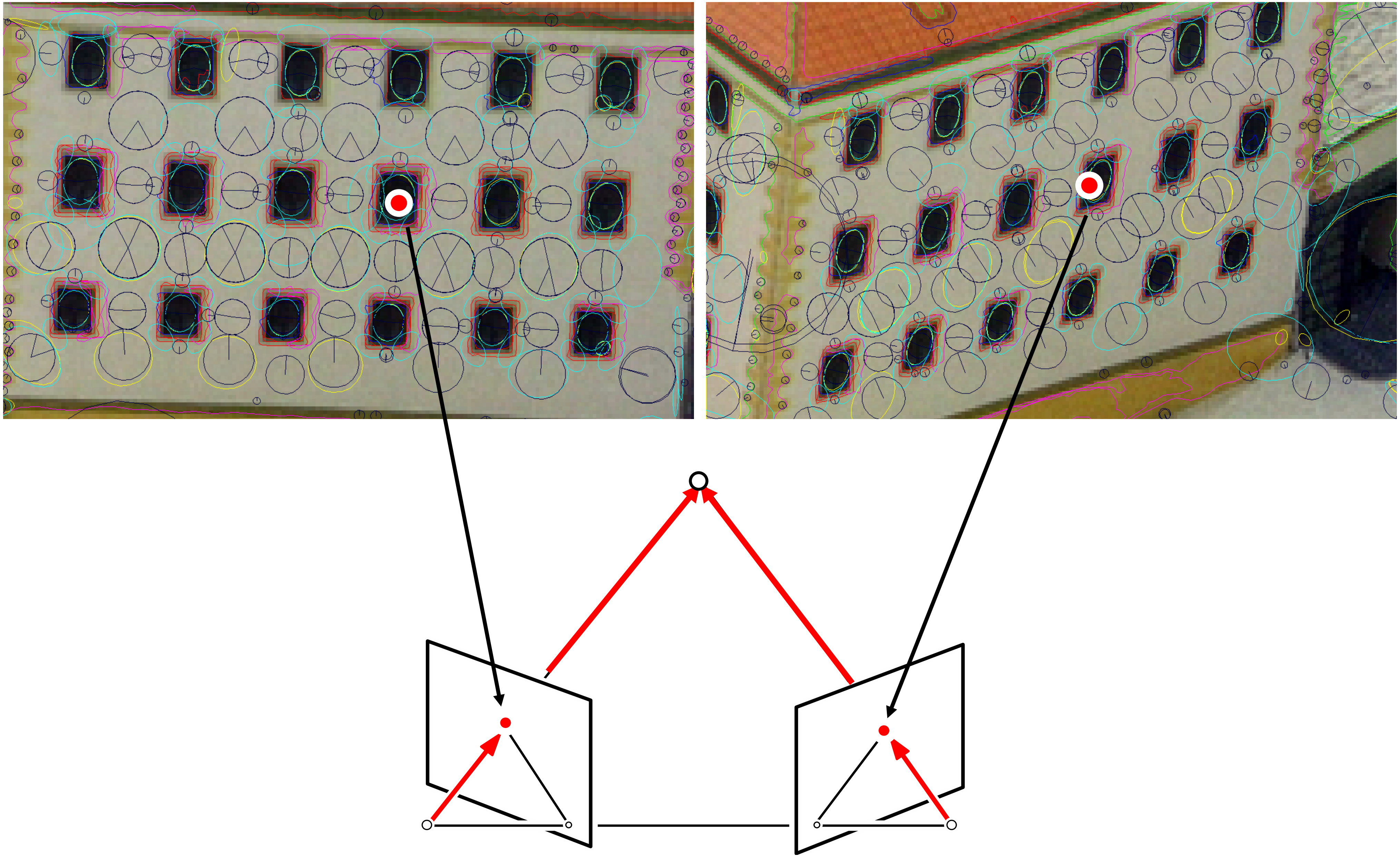
- Similar objects (circles, rectangles) … tentative matches
- Some are correct, some are wrong
- Optimization task:

Find the largest (large) subset of tentative matches
consistent with a valid two view geometry

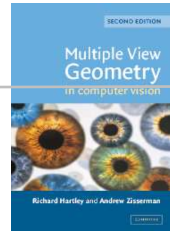
Valid two view geometry



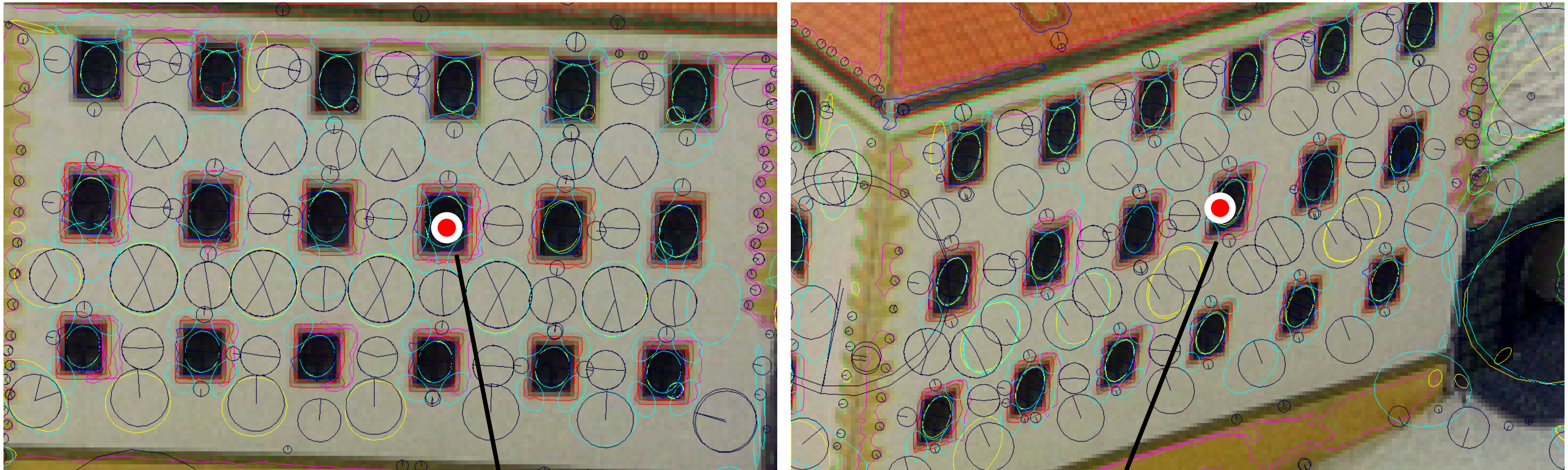
Epipolar constraint



Valid two view geometry

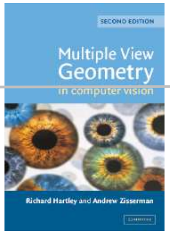


Epipolar constraint

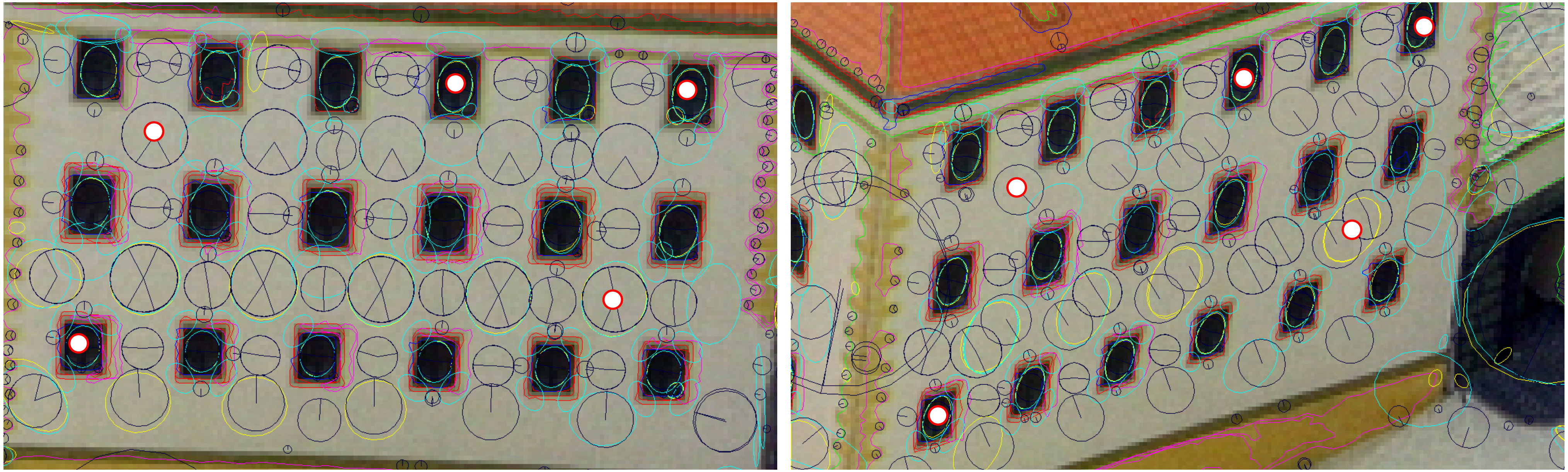


$$\mathbf{x}_2^\top \mathbf{F}\, \mathbf{x}_1 = 0$$

Valid two view geometry



Epipolar constraint

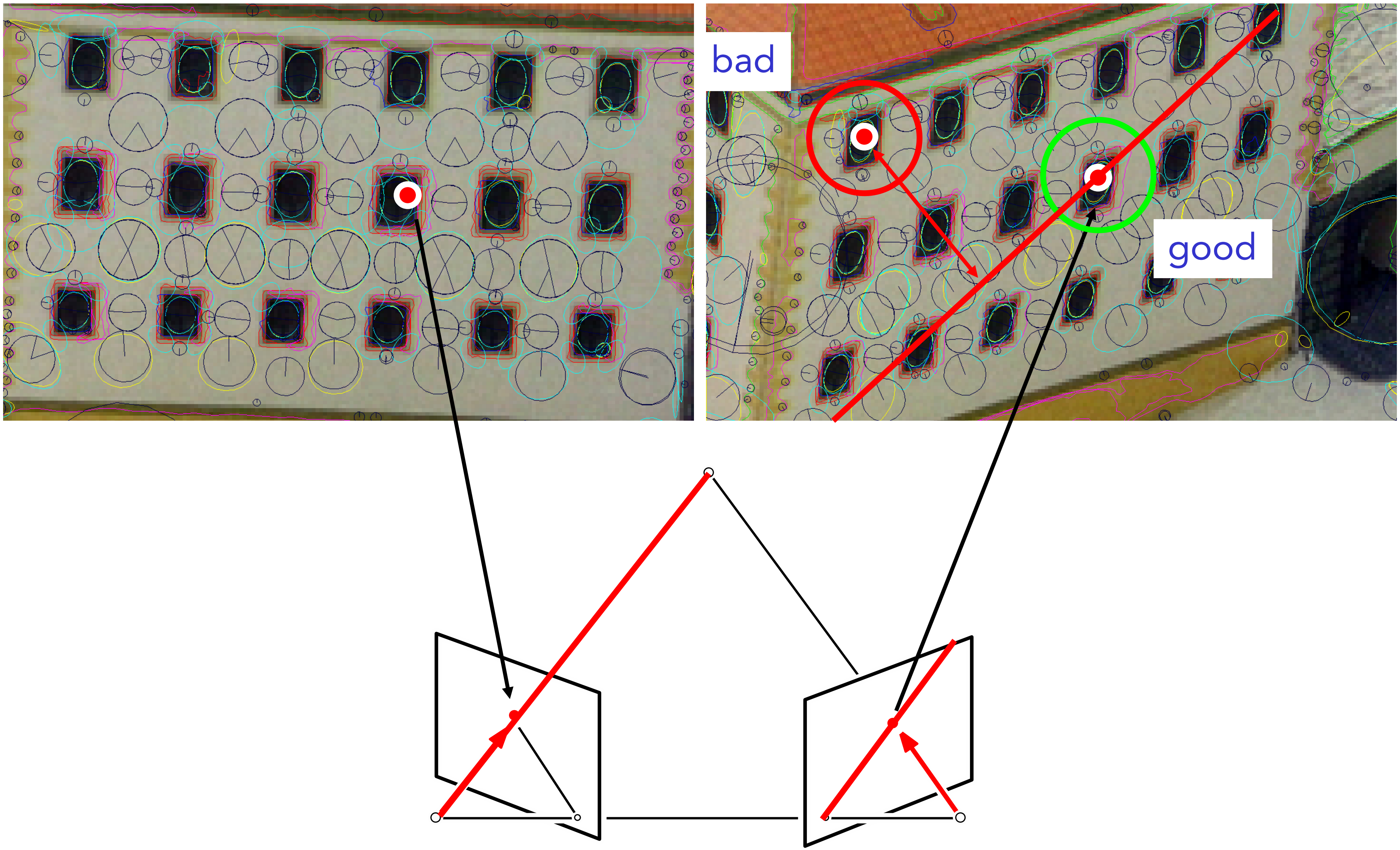


- F can be computed from 5 matches
- The best F is consistent with the largest subset

$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

Consistent two view geometry

Matching constraint



Optimization scheme = RANSAC

Enumerating all subsets replaced by checking only some of them

RANdom SAmpling Consensus

1. Generate random 5-tuples of matches
2. Compute F by solving $\mathbf{x_2}^\top \mathbf{F}\,\mathbf{x_1} = 0$
3. Count the number of good matches

Return the largest set of good matches

Why to be fast?

- Many samples needed to be sure to find a good sample!

To find a gross-error-free sample with 95% probability we have to try at least the following number of samples:

Gross error contamination ratio [%]

Sample size

	15%	20%	30%	40%	50%	70%
2	132	73	32	17	10	4
4	5916	1871	368	116	46	11
7	$1.75 \cdot 10^6$	$2.34 \cdot 10^5$	$1.37 \cdot 10^4$	1827	382	35
8	$1.17 \cdot 10^7$	$1.17 \cdot 10^6$	$4.57 \cdot 10^4$	4570	765	50
12	$2.31 \cdot 10^{10}$	$7.31 \cdot 10^8$	$5.64 \cdot 10^6$	$1.79 \cdot 10^5$	$1.23 \cdot 10^4$	215
18	$2.08 \cdot 10^{15}$	$1.14 \cdot 10^{13}$	$7.73 \cdot 10^9$	$4.36 \cdot 10^7$	$7.85 \cdot 10^5$	1838
30	∞	∞	$1.35 \cdot 10^{16}$	$2.60 \cdot 10^{12}$	$3.22 \cdot 10^9$	$1.33 \cdot 10^5$
40	∞	∞	∞	$2.70 \cdot 10^{16}$	$3.29 \cdot 10^{12}$	$4.71 \cdot 10^6$

Solving time: **micro-mili** seconds

How to be fast?

How to be fast?

1. Specialized solving methods

2. Assume generic data

3. Use tricks, optimize, hard code, …

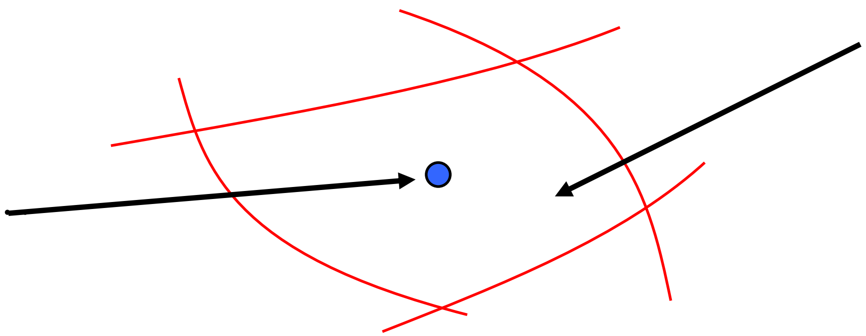
Many problems are generic

Solvers do not (much) differ from one problem o another.

→ Solver is made out by solving a single concrete system and then used on other systems

→ This works around generic solutions

a polynomial system used to build the solver . .



. . . it works here

parameter space

Strategy of fast solving

Offline phase (may be slow)

1. Fabricate a concrete generic example of a polynomial system (generating 0-dim radial ideal I)
2. Analyze the system by a generic method (Macaulay2, FGb, …) to get the degree, (standard monomial) basis in R/I, …
3. Create an elimination template for constructing a multiplication matrix $\mathtt{M}_f$ of multiplication by a suitable polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ (an unknown) in a finite-dimensional factor ring $A = \mathbb{C}[x_1, \ldots, x_n]/I$.
4. Implement efficiently in floating points, optimize, test, … (vary ordering, basis selection, …)

Strategy of fast solving

Online (must be fast)

1. Fill the elimination template to get matrix $\mathtt{M}_f$
2. Solve numerically by finding eigenvectors of $\mathtt{M}_f$
   (or get a univariate poly and use real root bracketing)

Offline: Fabricate a concrete generic example

1. An easy example

General problem		**Specific instance** $(\mathbb{Z}_7)$
$\begin{cases} f_1 = x^2 + y^2 - 1 = 0 \\ f_2 = x + ay + b = 0 \end{cases}$	$\Rightarrow$	$\begin{cases} f_1 = x^2 + y^2 - 1 = 0 \\ f_2 = x + 2y - 2 = 0 \end{cases}$

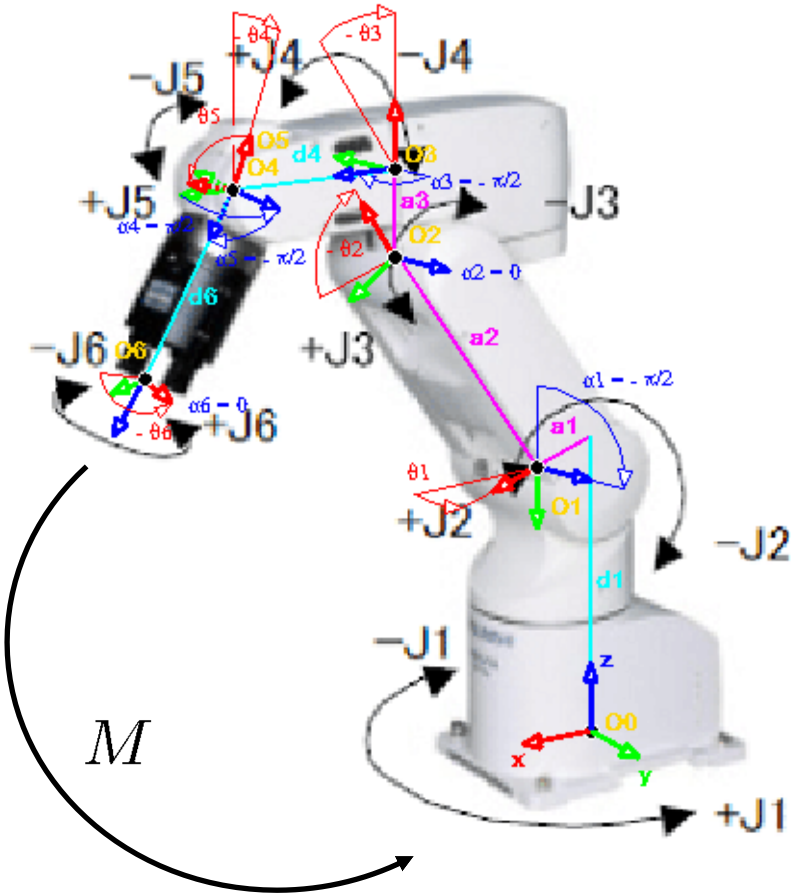
RAND

Offline: Fabricate a concrete generic example

2. A (not difficult) example (Inverse Kinematic Task in robotics)

   Given M find c_i, s_i (sin & cos of controlled angles)

$$M = M_1^0(c_1, s_1)\, M_2^1(c_2, s_2)\, M_3^2(c_3, s_3)\, M_4^3(c_4, s_4)\, M_5^4(c_5, s_5)\, M_6^5(c_6, s_6)$$



$$M_i^{i-1} = \begin{bmatrix} c_i & -s_i\, p_i & s_i\, q_i & a_i\, c_i \\ s_i & c_i\, p_i & -c_i\, q_i & a_i\, s_i \\ 0 & q_i & p_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$c_1^2 + s_1^2 = 1 \qquad c_4^2 + s_4^2 = 1$$
$$c_2^2 + s_2^2 = 1 \qquad c_5^2 + s_5^2 = 1$$
$$c_3^2 + s_3^2 = 1 \qquad c_6^2 + s_6^2 = 1$$

- M must contain a rotation matrix to get a consistent system.
- A rational rotation must be constructed (no difficult)

Offline: Fabricate a concrete generic example

3. Hard cases exist too

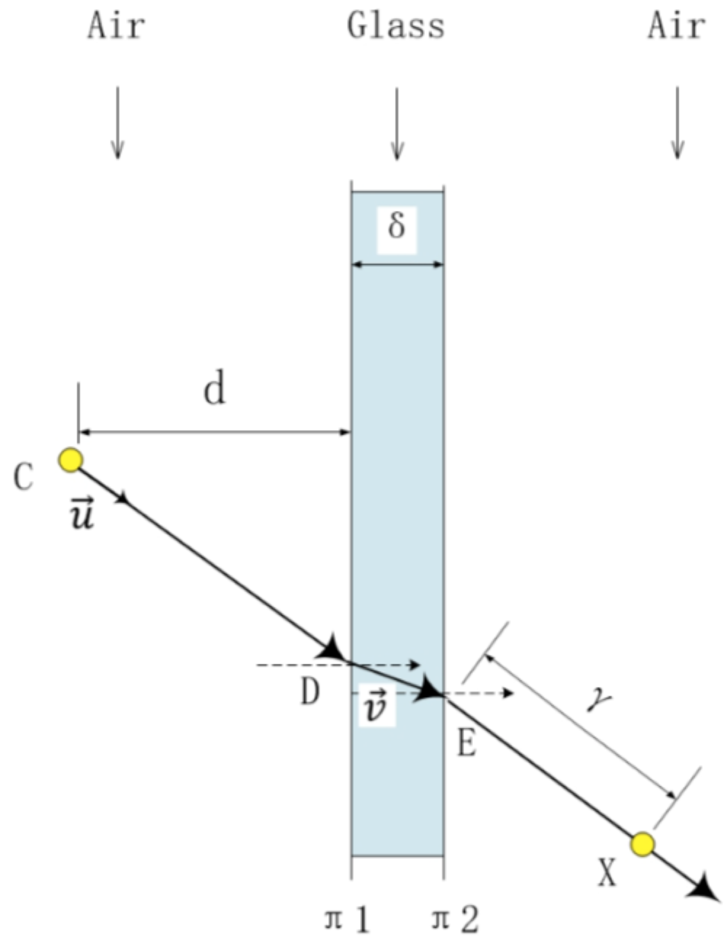


Figure 3.1: Model of the planar glass and projection geometry

A more general parametric systems:

Do we need to find a rational point on a variety?

- Hard?
- When possible/impossible?
- Other options (numerical) if impossible?

Offline: Analyze the system

Specific instance ($\mathbb{Z}_7$)

$$\begin{cases} f_1 = x^2 + y^2 - 1 = 0 \\ f_2 = x + 2y - 2 = 0 \end{cases}$$

```
Macaulay2, version 1.12
-- Ring
i1 : R = ZZ/7[x,y]

-- An instance
i2 : F = matrix({{x^2+y^2-1},{x+2*y-2}})
i3 : I = ideal(F)

-- The dimension and the degree of V(I)
i4 : dim(I), degree(I)

o4 = (0, 2)

-- The standard monomial basis of A = R/I
i5 : A = R/I
i6 : B = basis(A)

o6 = | y 1 |
```

Offline: Create an elimination template

$$y\;B = \boxed{M}\;B + \overbrace{T\qquad F}^{\in I}$$

$$y\begin{pmatrix} y \\ 1 \end{pmatrix} = \boxed{\begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix}}\begin{pmatrix} y \\ 1 \end{pmatrix} + \begin{bmatrix} 3 & -3\,x - y + 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x^2 + y^2 - 1 \\ x + 2\,x + 2 \end{pmatrix}$$

$$\begin{array}{cccccccc}
x^2 & xy & x & y^2 & y & 1 & & \\
\hline
1 & & & 1 & & -1 & f_1 & 1 \cdot (x^2 + y^2 - 1) \\
1 & a & b & & & & x f_2 & x \cdot (x + a\,x + b) \\
 & 1 & & a & b & & y f_2 & y \cdot (x + a\,x + b) \\
 & & 1 & & a & b & f_2 & 1 \cdot (x + a\,x + b) \\
\hline
\end{array}$$

Gauss-Jordan Elimination

$$\begin{array}{cccccc}
x^2 & xy & x & y^2 & y & 1 \\
\hline
1 & & & & \bullet & \bullet \\
 & 1 & & & \bullet & \bullet \\
 & & 1 & & \bullet & \bullet \\
 & & & \boxed{1} & \boxed{-m_{11}} & \boxed{-m_{12}} \\
\hline
\end{array}$$

Offline: Create an elimination template

```
Macaulay2
-- Reduce y multiple of B by I
i9 : r = (y*B) % I
o9 = | 3y-2 y |

-- Multiplication matrix of y in R/I w.r.t. B
i10 : M = transpose((coefficients(r,Monomials=>B))_1)
o10 = {-1} | 3 -2 |
      {0}  | 1  0 |

-- Create the template for constructing M from F
-- y*B = M*B + T*F
i12 : T = transpose(transpose(y*B-M*B) // gens(I))
o12 = {-1} | 3 -3x-y+1 |
      {0}  | 0 0       |

-- Extract monomials from columns of T
i14 : m = apply(m,x->apply(x,x->(… (T) …))
o14 = {{1}, {1, y, x}}
```

Offline: Create an elimination template

```
-- Create the template for constructing M from F
-- y*B = M*B + T*F
i12 : T = transpose(transpose(y*B-M*B) // gens(I))
o12 = {-1} | 3 -3x-y+1 |
      {0}  | 0 0       |
```

`//` … does the magic

- Traces Groebner basis construction
- Reduces the template by the Groebner basis of the Syzygy module
- Often produces very efficient templates (no unnecessary rows) but not always
- When can be ``the optimal template" defined and found? (F5?)
- Other parameters important … e.g. basis in R/I selection

Online: Fill the template, get M, eigenvectors

New coefficients a, b

$$x^2 + y^2 - 1$$
$$x + a\,x + b$$

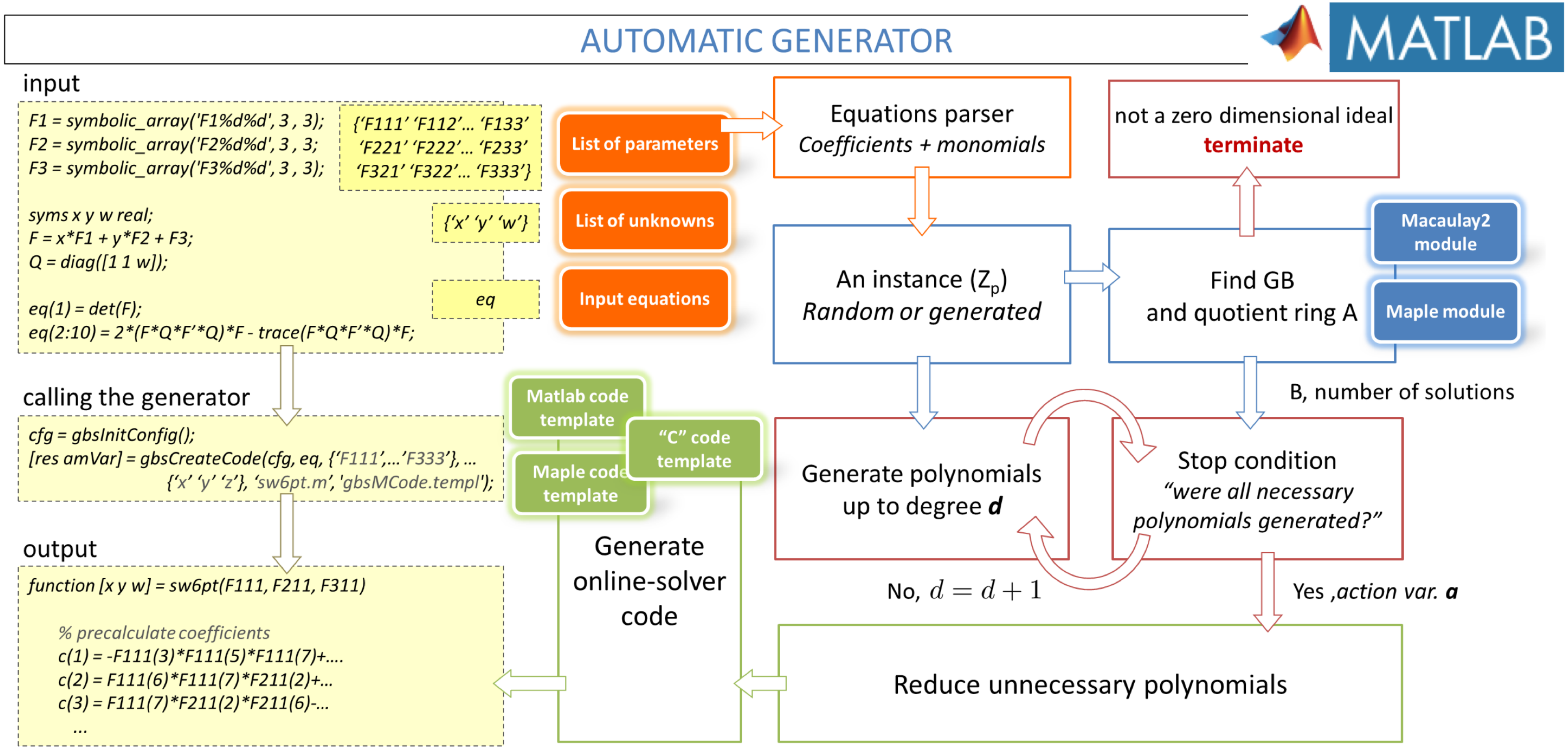
x^2	xy	x	y^2	y	1
1			1		-1
1	a	b			
	1		a	b	
		1		a	b

Gauss-Jordan Elimination

x^2	xy	x	y^2	y	1
1				$\bullet$	$\bullet$
	1			$\bullet$	$\bullet$
		1		$\bullet$	$\bullet$
			1	$-m_{11}$	$-m_{12}$

Compute solutions $v \simeq \begin{pmatrix} y \\ 1 \end{pmatrix}$ by $\lambda v = \begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix} v$

Automatic generator of ``minimal solvers"



- Z Kukelova, M Bujnak, T Pajdla.
  Automatic Generator of Minimal Problem Solvers. ECCV 2008.
- V Larsson, K Astrom, M Oskarsson.
  Efficient Solvers for Minimal Problems by Syzygy-Based Reduction. CVPR 2017.
- V Larsson, M Oskarsson, K Astrom, A Wallis, Z Kukelova, T Pajdla.
  Beyond Grobner Bases: Basis Selection for Minimal Solvers. CVPR 2018

Template construction optimization

- Criteria for the best template have not (yet) been clearly defined but
- Efficient templates are small and numerically robust
- R/I basis selection is important (for the strategy described)

Basis selection in $A = R/I$

- How to choose a good basis?
- Experiments with monomial orderings + more …

Basis selection in A = R/I

- Only a finitely many different Groebner bases (Groebner fan)

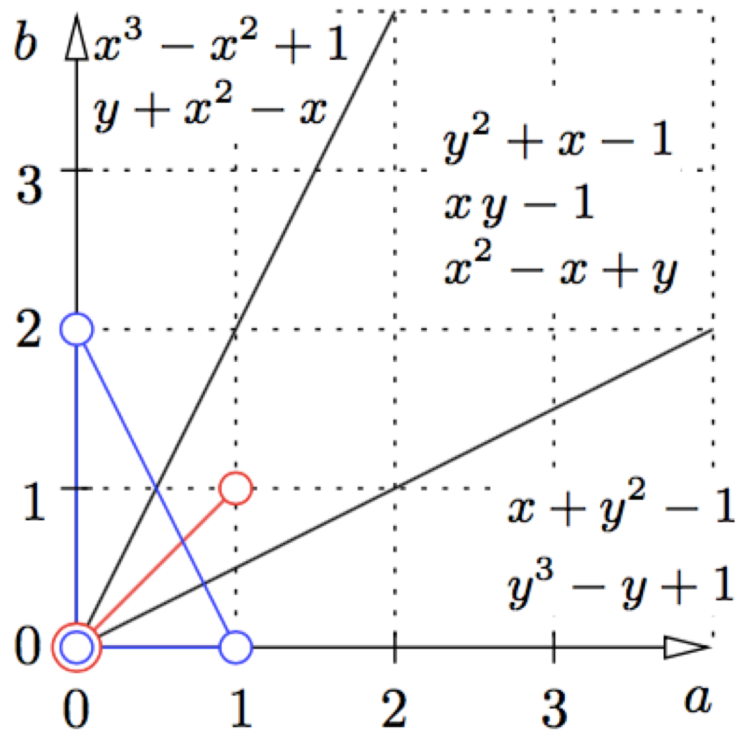
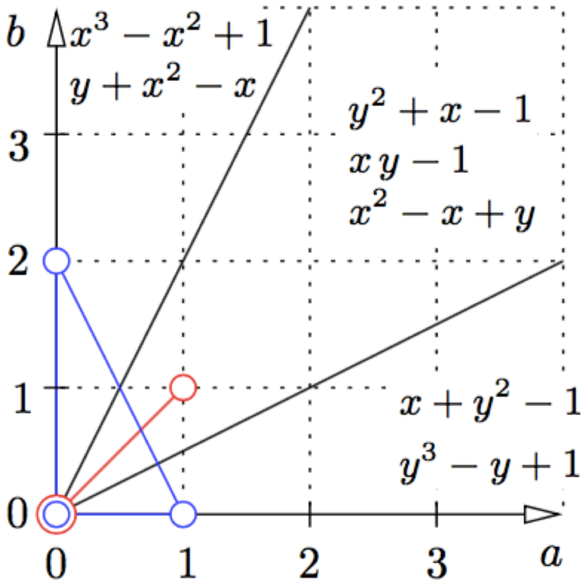


Figure 1. The Gröbner fan of the ideal $I = \langle x + y^2 - 1, x\,y - 1\rangle$ consists of three two-dimensional cones. For each cone, there is exactly one reduced Gröbner basis of I. All monomial orderings generated by all weight vectors from one cone give the same reduced Gröbner basis of I. Hence, there are exactly three different reduced Gröbner bases for I over all possible different monomial orderings.

Basis selection in A = R/I

- Generate standard monomial bases for all GB in the GB fan.
- Test them an choose the best (the smallest and stable) basis.
- Go beyond: Use a heuristic to sample other even better bases.



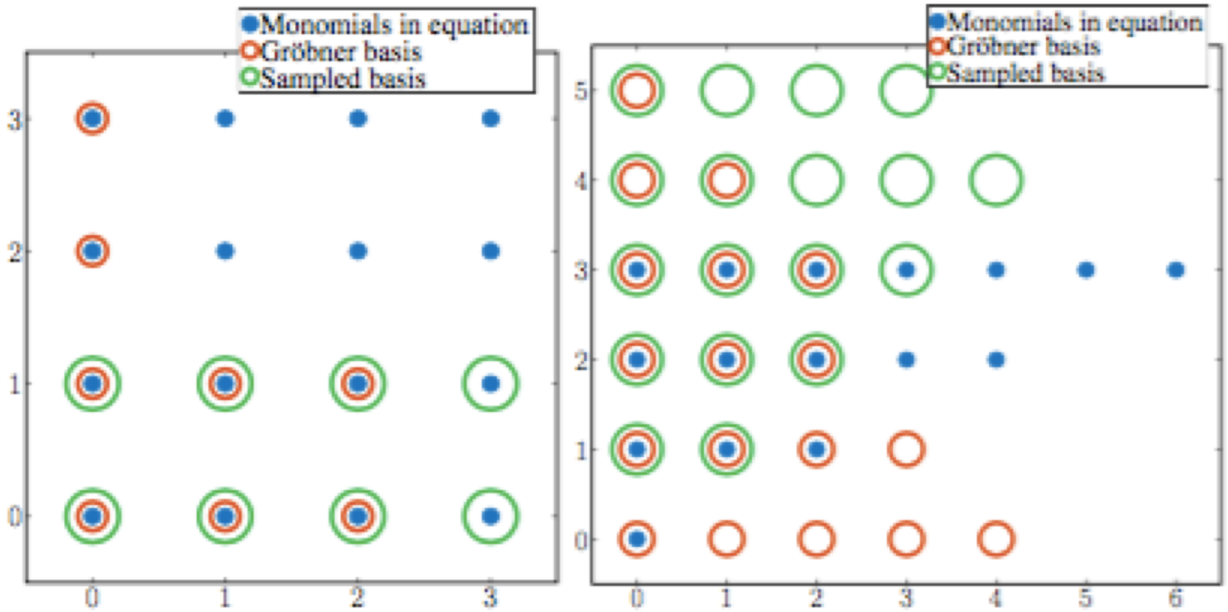


Figure 3. The figure shows the basis monomials for two example problems, namely 8pt rel. pose F+λ (left) and 3pt image stitching fλ+R+fλ (right). Both these problems have two variables, and for both these problems the proposed basis sampling scheme gives significantly smaller template compared to the Gröbner basis vari-

Basis selection in A = R/I

- There are dramatic differences

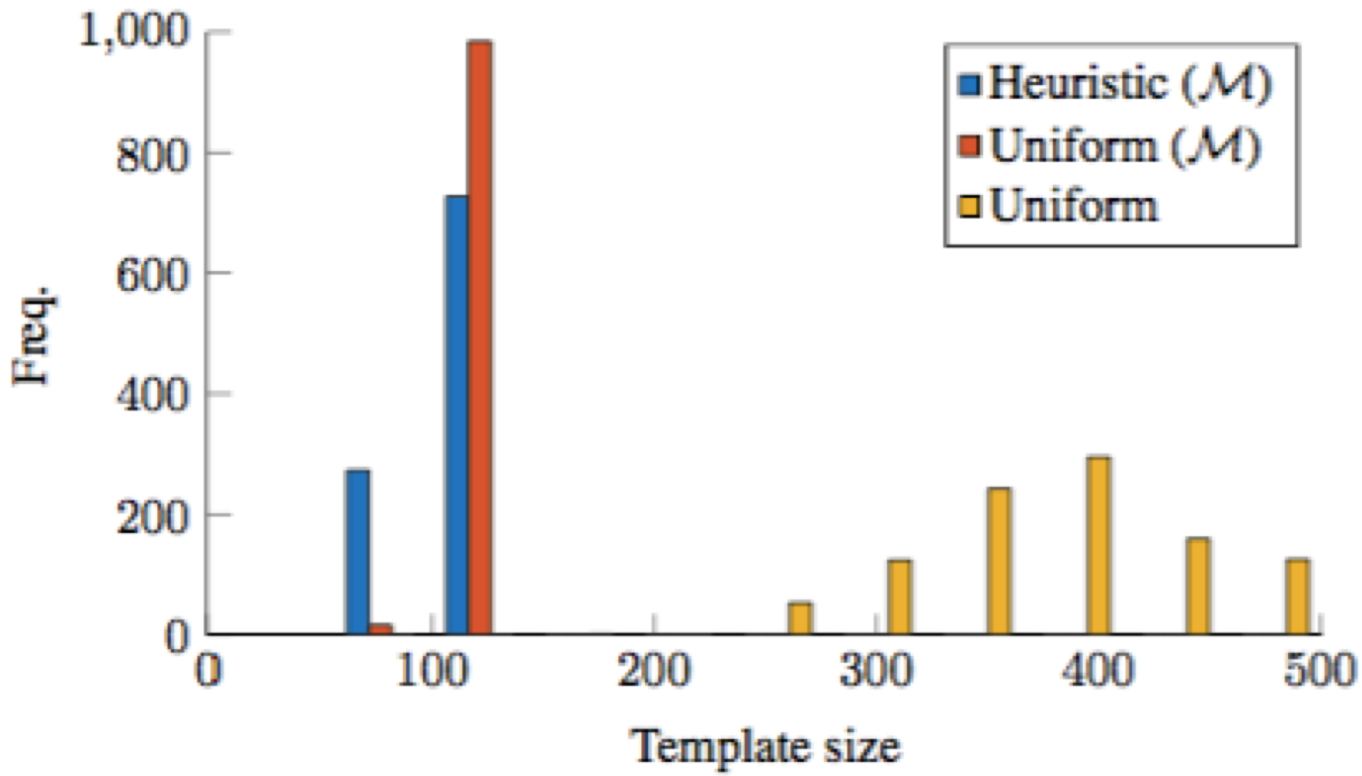


Figure 4. Template size (rows) for 1,000 randomly sampled bases for the P4Pfr formulation from Bujnak *et al.* [7].

Basis selection in A = R/I

- Many solvers improved

Problem	Author	Original	[29]	GFan+ [29]	(#GB)	Heuristic+[29]
Rel. pose F+λ 8pt	Kuang *et al.* [25]	12×24	11×20	11×20	(10)	$\mathbf{7 \times 16}$
Rel. pose E+f 6pt	Bujnak *et al.* [6]	21×30	21×30	$\mathbf{11 \times 20}$	(66)	$\mathbf{11 \times 20}$
Rel. pose f+E+f 6pt	Kukelova *et al.* [26]	31×46	31×50	31×50	(218)	$\mathbf{21 \times 40}$
Rel. pose E+λ 6pt	Kuang *et al.* [25]	48×70	34×60	34×60	(846)	$\mathbf{14 \times 40}$
Stitching $f\lambda$+R+$f\lambda$ 3pt	Naroditsky *et al.* [35]	54×77	48×66	48×66	(26)	$\mathbf{18 \times 36}$
Abs. Pose P4Pfr	Bujnak *et al.* [7]	136×152	140×156	$\mathbf{54 \times 70}$	(1745)	$\mathbf{54 \times 70}$
Rel. pose λ+E+λ 6pt	Kukelova *et al.* [26]	238×290	149×205	-	?	$\mathbf{53 \times 115}$
Rel. pose λ_1+F+λ_2 9pt	Kukelova *et al.* [26]	179×203	165×200	$\mathbf{84 \times 117}$	(6896)	$\mathbf{84 \times 117}$
Rel. pose E+$f\lambda$ 7pt	Kuang *et al.* [25]	200×231	181×200	$\mathbf{69 \times 90}$	(3190)	$\mathbf{69 \times 90}$
Rel. pose E+$f\lambda$ 7pt (elim. λ)	-	-	52×71	37×56	(332)	$\mathbf{24 \times 43}$
Rel. pose E+$f\lambda$ 7pt (elim. $f\lambda$)	Kukelova *et al.* [27]	$\mathbf{51 \times 70}$	$\mathbf{51 \times 70}$	$\mathbf{51 \times 70}$	(3416)	$\mathbf{51 \times 70}$
Abs. pose quivers	Kuang *et al.* [22]	372×386	216×258	-	?	$\mathbf{81 \times 119}$
Rel. pose E angle+4pt	Li *et al.* [32]	270×290	266×329	-	?	$\mathbf{183 \times 249}$
Abs. pose refractive P5P	Haner *et al.* [17]	280×399	240×324	$\mathbf{157 \times 246}$	(8659)	240×324

Table 1. Size of the elimination templates for some minimal problems. For the relative pose problems unknown radial distortion is denoted with λ and unknown focal length with f, and the position describes which camera it refers to. The table shows the original template size from the author, the template size found using the method from [29] (GRevLex basis), the template size from doing an exhaustive search over Gröbner bases (Section 2.2) and the random sampling approach (Section 3.1). Missing entries are when the Gröbner fan computation took longer than 12 hours.

Minimal Problems in Computer Vision

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